Studies on $\mathbf{H}_\infty$ Filtering Problems for Linear Discrete-Time Systems

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ABSTRACT

This thesis is concerned with the $\mathbf{H}_\infty$ filtering problem for linear discrete-time systems. The $\mathbf{H}_\infty$ filtering problem is a state estimation problem of minimizing the maximum energy in the estimation error over all possible disturbance trajectories. The state estimation based on the $\mathbf{H}_\infty$ criterion is valid when there exists a significant uncertainty in the disturbance statistics. This thesis consists of mainly two parts.

The first part considers the infinite-horizon problem for a time-invariant system. We provide a complete solution to the infinite-horizon $\mathbf{H}_\infty$ filtering problem for time-invariant systems from the viewpoint of model matching in the frequency domain. The set of all $\mathbf{H}_\infty$ filters is characterized in terms of a positive semi-definite stabilizing solution of the $\mathbf{H}_\infty$ algebraic Riccati equation (ARE).

The free parameter contained in the $\mathbf{H}_\infty$ filter can be used for achieving an additional design specification as well as the $\mathbf{H}_\infty$ error bound. In the case where the system is subject to step and/or periodic disturbances, the state estimates may be degraded by the biases or periodic fluctuations due to these disturbances. In order to attenuate these undesirable effects of these disturbances, the transfer functions from the disturbances to the estimation error must be zero at certain points on the unit circle of the complex plane. Based on the Nevanlinna-Pick interpolation theory, we propose a method for adjusting the free parameter so that the boundary constraints on the unit circle are satisfied.

Since the $\mathbf{H}_\infty$ filter is characterized by a positive semi-definite stabilizing solution of the $\mathbf{H}_\infty$ ARE, the estimation performance of the $\mathbf{H}_\infty$ filter is dependent on the properties of the $\mathbf{H}_\infty$ ARE. We derive a lower bound of the $\mathbf{H}_\infty$ error bound γ for which there exists a stabilizing solution of the $\mathbf{H}_\infty$ ARE, and show the monotonicity and convexity of the stabilizing solution with respect to γ. Furthermore, based on the above properties of the stabilizing solution of the $\mathbf{H}_\infty$ ARE, we study the behavior of the set of all $\mathbf{H}_\infty$ filters with respect to the change of γ. It turns out that the degree of freedom of the $\mathbf{H}_\infty$ filter decreases at the optimum under a certain condition.

In the second part, the finite-horizon problem for a time-varying system is studied. Since the $\mathbf{H}_\infty$ norm is the $\mathbf{L}_2$ induced norm, the $\mathbf{H}_\infty$ filtering algorithm has a certain

minimax properties. In order to understand this aspect of the $\mathbf{H}_\infty$ filtering problem, it is essential to exploit the game theoretic approach in the time-domain. It is shown that the solutions to the minimax filtering and prediction problems are identical to the central $\mathbf{H}_\infty$ filter and the $\mathbf{H}_\infty$ one-step predictor, respectively. The worst-case disturbance maximizing the energy in the estimation error is also derived.

By using the Riccati difference equations (RDEs), we compare the performances of the $\mathbf{H}_\infty$ and Kalman filters in the case where the disturbances are zero mean Gaussian white noises. The relation between the prescribed $\mathbf{H}_\infty$ error bound γ and the estimation performance of the central $\mathbf{H}_\infty$ filter is examined based on the monotonicity of the $\mathbf{H}_\infty$ RDE. For time-invariant systems, the connection between the finite and infinite horizon $\mathbf{H}_\infty$ filtering problems is made clear by showing the convergence of the solution of $\mathbf{H}_\infty$ RDE. We also derive a solution to the $\mathbf{H}_\infty$ fixed-lag smoothing problem based on the result on the $\mathbf{H}_\infty$ filtering problem.

Finally, we discuss the existence of a saddle point solution to the stochastic minimax filtering and prediction problems. It turns out that the minimizer's saddle point policies are identical to the central $\mathbf{H}_\infty$ filter and the $\mathbf{H}_\infty$ predictor. These results provide alternative interpretations of these $\mathbf{H}_\infty$ state estimators and a justification of the application of the $\mathbf{H}_\infty$ state estimators to the stochastic system with unknown disturbance statistics.

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Notations and Definitions

$(\,\cdot\,)^{\mathrm{T}}$	Transpose of a matrix
$(\,\cdot\,)^{\mathrm{H}}$	Complex conjugate transpose of a matrix
$(\,\cdot\,)^{\#}$	Moor-Penrose pseudo-inverse
$\|\,\cdot\,\|$	Norm of a vector and matrix (Euclidean norm for a vector, the largest singular value for a matrix)
$\|x\|_Q$	$:= (x^{\mathrm{T}}Qx)^{1/2}$
$E\{\,\cdot\,\}$	Expectational operation
$\mathrm{Tr}(\,\cdot\,)$	Trace of a square matrix
$\lambda_{\max}(\,\cdot\,)$	The largest eigenvalue
$\lambda_{\min}(\,\cdot\,)$	The smallest eigenvalue
$\mathrm{diag}[\cdots]$	Block diagonal matrix formed from the arguments
$\mathrm{Im}(\,\cdot\,)$	Image or range space of a matrix
$\mathrm{Ker}(\,\cdot\,)$	Kernel or null space of a matrix
I_n	The $n \times n$ identity matrix. The subscript may be omitted when it is irrelevant.
$\mathcal{N}(m, R)$	Gaussian distribution with mean m and covariance R
$\mathbf{R}$	The set of all real numbers
$\mathbf{R}^n$	The set of all n-dimensional real vectors
$\mathbf{R}^{m\times n}$	The set of all $m \times n$ real matrices
$\mathbf{C}$	The set of all complex numbers
$\mathbf{C}^n$	The set of all n-dimensional complex vectors
$\mathbf{C}^{m\times n}$	The set of all $m \times n$ complex matrices
σ	The shift operator
$\mathbf{H}_\infty$	The set of all proper stable transfer matrices
$\mathbf{RL}_\infty$	The set of all proper real rational transfer matrices which have no poles on the unit circle
$\mathbf{RH}_\infty$	The set of all proper stable real rational transfer matrices
$\mathbf{GH}_\infty$	The set of all unimodular transfer matrices
$\mathbf{BH}_\infty$	The set of all transfer matrices in $\mathbf{RH}_\infty$ whose $\mathbf{H}_\infty$ norms are less than γ.
$\overline{\mathbf{B}}\mathbf{H}_\infty$	The set of all transfer matrices in $\mathbf{RH}_\infty$ whose $\mathbf{H}_\infty$ norms are less than or equal to γ.
$X^{m\times n}$	The set of all $m \times n$ transfer matrices in the function space X
$\mathbf{L}_2$	The set of all square summable functions
$\mathbf{L}_2[0, N]$	The set of all signals which are square summable over the interval $[0, N]$
$G^{\sim}(\sigma)$	The parahermitian conjugate of $G(\sigma)$, namely $G^{\mathrm{T}}(\sigma^{-1})$
J_{pq}	$\mathrm{diag}\,[\,I_p \quad -\gamma^2 I_q\,]$ for a given constant $\gamma > 0$

A transfer matrix in the state-space data is written as

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(\sigma I - A)^{-1}B + D$$

The $\mathbf{L}_2$ and $\mathbf{L}_2[0, N]$ norms of a function x_k are respectively defined by

$$\|x\|_2 = \left(\sum_{k=0}^{\infty} x_k^{\mathrm{T}} x_k\right)^{1/2} \quad \text{and} \quad \|x\|_2 = \left(\sum_{k=0}^{N} x_k^{\mathrm{T}} x_k\right)^{1/2}$$

The $\mathbf{H}_2$ norm of a transfer matrix $G(\sigma)$ is defined by

$$\|G\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Tr}\{G^{\mathrm{H}}(e^{j\omega})G(e^{j\omega})\}d\omega\right)^{1/2}$$

The $\mathbf{L}_\infty$ and $\mathbf{H}_\infty$ norms are defined by

$$\|G\|_\infty = \sup_{\omega\in\mathbf{R}} \|G(e^{j\omega})\|$$

Acronyms

ARE	Algebraic Riccati Equation
RDE	Riccati Difference Equation

Chapter 1

Introduction

1. Kalman Filter and Minimax Filters

The filtering problem involves estimating the states of a system using the past noisy measurements. Since the publication of the fundamental papers by R. E. Kalman [24],[25], Kalman filtering theory based on the least-squares ($\mathbf{H}_2$-optimal) error criterion has been deeply entrenched in the control and signal processing theories and their applications for more than three decades (see, for example, [26],[41]). When the noise disturbances are white noise processes and their spectral densities are exactly known, Kalman filter offers the optimal state estimation algorithm in the least-squares and minimum-variance senses that $\|e\|_2$ and $E\{\|e_k\|^2\}$ are minimized, where e_k denotes the estimation error. However, it is difficult to know the exact stochastic properties of the disturbances *a priori*. In this case, the state estimates based on the least-squares criterion may be degraded by the uncertainty of the disturbance statistics.

To cope with this difficulty, a number of researches on the robust filtering have been reported. One of the major approaches to the robust filtering under uncertain disturbance statistics is the minimax filtering based on the game theory. Mintz [33] and Krener [30] showed that the Kalman filter has the minimax property for the following pointwise optimization problem:

$$\min_{\text{filter}} \max_{d_k \in \mathbf{L}_2[0,N]} \left\{ \|e_N\|^2 - \sum_{k=0}^{N} \|d_k\|^2 \right\}$$

where d_k denotes the disturbance and k is the time step. Moreover, this minimax state estimation problem has been recently reconsidered by Basar in the prediction and smoothing cases [2]. As a different minimax approach to the design of a robust Kalman filter, Poor and Looze considered the minimax problem where the disturbances are known to be white noises, while their covariances are unknown and belong to certain compact convex sets [37].

This thesis addresses a new minimax approach to the robust filtering problem based on the $\mathbf{H}_\infty$ error criterion which has received great interest in the robust control theory.

2. $\mathbf{H}_\infty$ Error Criterion

In the last several years, the $\mathbf{H}_\infty$ control theory has brought a remarkable breakthrough in the field of robust control. The interested readers should refer to the text books such as [11], [17] and [42]. The two Riccati formula for the state-space solution to the standard $\mathbf{H}_\infty$ control problem was first derived by Doyle *et al.* [8], and thereafter many techniques for solving this problem were reported in the literature (see e.g. [15],[28],[42]).

This thesis addresses a new minimax filtering problem based on $\mathbf{H}_\infty$ error criterion. That is, we employ the $\mathbf{H}_\infty$ norm of the error dynamics as a measure of the estimation errors. Since the $\mathbf{H}_\infty$ norm is the $\mathbf{L}_2$ induced norm, i.e. the maximum energy in the output signal over all possible exogenous input trajectories, the filtering algorithm based on the $\mathbf{H}_\infty$ criterion possesses a minimax property. Thus, $\mathbf{H}_\infty$ criterion is valid in the case where there exists a significant uncertainty in the spectrum density of the exogenous disturbance [56]. As shown below, the $\mathbf{H}_\infty$ filtering problem is different from the minimax problems mentioned in the previous section, because it involves the minimization of the accumulated estimation error rather than the pointwise minimization of the estimation error.

We here briefly review the minimax aspect of the $\mathbf{H}_\infty$ filtering problem. We consider the linear time-invariant case for simplicity. Let $T_{ed}(\sigma)$ be the transfer matrix from the disturbance d_k to the estimation error e_k. The z-transform of e_k is then given by $e(\sigma) = T_{ed}(\sigma)d(\sigma)$. We first assume that d_k is an arbitrary deterministic $\mathbf{L}_2$ signal. If the filter, denoted by $T_f(\sigma)$, is designed to stabilize $T_{ed}(\sigma)$, e_k is also an $\mathbf{L}_2$ signal. By Parseval's theorem, we see that

$$\begin{aligned}\|e\|_2^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d^{\mathrm{H}}(e^{j\omega})T_{ed}^{\mathrm{H}}(e^{j\omega})T_{ed}(e^{j\omega})d(e^{j\omega})d\omega \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} d^{\mathrm{H}}(e^{j\omega})d(e^{j\omega})\|T_{ed}(e^{j\omega})\|^2 d\omega \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} d^{\mathrm{H}}(e^{j\omega})d(e^{j\omega})\|T_{ed}\|_\infty^2 d\omega = \|T_{ed}\|_\infty^2\|d\|_2^2\end{aligned}$$

Hence, the $\mathbf{H}_\infty$ filtering problem of designing a filter $T_f(\sigma)$ satisfying $\|T_{ed}\|_\infty < \gamma$ for a given $\gamma > 0$ has the following minimax property.

$$\inf_{T_f\,\text{stabilizing}\,T_{ed}}\ \sup_{d_k\in\mathbf{L}_2}\ (\|e\|_2^2 - \gamma^2\|d\|_2^2) < 0$$

Let $\mathcal{P}$ denote the set of all second-order stationary processes. Suppose that the disturbance d_k belongs to $\mathcal{P}$. Then, the estimation error e_k also belongs to $\mathcal{P}$ if $T_{ed}(\sigma) \in \mathbf{RH}_\infty$. The auto-correlation matrix of d_k is defined by $R_d(\tau) = E\{d_{k+\tau}d_k^{\mathrm{T}}\}$. The Fourier transformation of $R_d(\tau)$, denoted by $S_d(\omega)$, is called the power spectral density matrix of d_k, namely,

$$S_d(\omega) = \sum_{\tau=-\infty}^{\infty} R_d(\tau)e^{-j\omega\tau}$$

Similarly, we define $S_e(\omega)$ as the power spectral density matrix of e_k. It is easy to verify that

$$S_e(\omega) = T_{ed}(e^{j\omega})S_d(\omega)T_{ed}^{\mathrm{H}}(e^{j\omega})$$

We thus obtain

$$\mathrm{Tr}S_e(\omega) \le \|T_{ed}(e^{j\omega})\|^2\mathrm{Tr}S_d(\omega) \le \|T_{ed}\|_\infty^2\mathrm{Tr}S_d(\omega)$$

We also easily see by the inverse Fourier transform that

$$E\{d_k^{\mathrm{T}}d_k\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Tr}S_d(\omega)d\omega,\quad E\{e_k^{\mathrm{T}}e_k\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Tr}S_e(\omega)d\omega$$

Therefore, the $\mathbf{H}_\infty$ filtering problem has a minimax property for the stochastic noise disturbances, too.

$$\inf_{T_f\,\text{stabilizing}\,T_{ed}}\ \sup_{d_k\in\mathcal{P}}\left[E\{\|e_k\|^2\} - \gamma^2E\{\|d_k\|^2\}\right] < 0$$

3. $\mathbf{H}_\infty$ Filtering Problem

The $\mathbf{H}_\infty$ filtering problem was first addressed based on the polynomial approach for a discrete-time system [18]. This approach employs Kawakernaak's technique [31] which translates the $\mathbf{H}_\infty$ optimization problem to a certain $\mathbf{L}_2$ optimization problem. The polynomial approach was also applied to the fixed-lag smoothing problem [19].

A state-space approach to the $\mathbf{H}_\infty$ filtering problem was first studied for the continuous-time case [3]. As well known, the bounded real lemma (BRL) is one of the important tools for solving $\mathbf{H}_\infty$ optimization problems in the state-space setting. Based on the BRL and Lagrange multiplier technique, Bernstein *et al.* [3] considered the problem of minimizing the upper bound on the $\mathbf{L}_2$ norm of the estimation error while maintaining the $\mathbf{H}_\infty$ norm bound. Shaked [39] also provided a state-space solution for a linear stationary process based on the duality between estimation and control. Nagpal and Khargonekar [36] employed a time domain approach based on the game theoretic LQ optimization technique in order to derive necessary and sufficient conditions for the existence of solutions to both finite and infinite horizon $\mathbf{H}_\infty$ filtering problems. They also provided a solution to the $\mathbf{H}_\infty$ fixed-interval smoothing problem, and showed that the $\mathbf{H}_\infty$ smoother is optimal in the $\mathbf{H}_2$ sense [36]. Yaesh and Shaked [51],[54] gave another game theoretic interpretations of the $\mathbf{H}_\infty$ filter. Recently, the mixed $\mathbf{H}_2/\mathbf{H}_\infty$ filtering problem was solved using a convex optimization technique by Khargonekar and Rotea [27]. Moreover, for the finite-horizon problems, Uchida and Fujita [47] showed that the central $\mathbf{H}_\infty$ filter and $\mathbf{H}_\infty$ smoother minimize the exponential quadratic cost. It may be noted that a parametrization of all $\mathbf{H}_\infty$ filters is given as a solution to the special case of the standard $\mathbf{H}_\infty$ control problem [8]. However, this result cannot be directly applied to unstable systems because the control problem requires the internal stability of the closed-loop system, which cannot be satisfied in the filtering problem for unstable systems. Takaba and Katayama [46] provided a parametrization of all $\mathbf{H}_\infty$ filters based on the Nehari-type model matching technique.

The results for the discrete-time case in the state-space setting almost parallel the continuous-time case. The mixed $\mathbf{H}_2/\mathbf{H}_\infty$ one-step prediction problem was solved by Haddad *et al.* [20]. The BRL was also applied to the $\mathbf{H}_\infty$ filtering and $\mathbf{H}_\infty$ one-step prediction problems by Yaesh and Shaked [52]. Moreover, Yaesh and Shaked [55] provided a game theoretic interpretation of the $\mathbf{H}_\infty$ one-step predictor, which is discrete-time counterpart of the result of [54]. However, the above works assumed that the state estimator has a so-called 'full-order observer' structure. Thus, a complete parametrization of all $\mathbf{H}_\infty$ filters has not been given. Moreover, unlike the continuous-time case, two full-order observer structures, namely, a filter and a one-step predictor, are possible in the discrete-time case. Thus, the solutions to the filtering and prediction problems were derived from the different problem formulations. For the finite-horizon case, Fujita *et al.* [12] recently derived a necessary and sufficient condition for the existence of an $\mathbf{H}_\infty$ filter based on the completing the squares and conjugate point arguments without assuming the observer structure. They also demonstrated the superiority of the $\mathbf{H}_\infty$ filter to the Kalman filter in a visual tracking system.

4. Overview of the Thesis

This thesis mainly consists of two parts. In Chapters 2–4, we study the $\mathbf{H}_\infty$ filtering problem for time-invariant systems in the frequency-domain setting. Chapters 5–7 are concerned with the finite time horizon $\mathbf{H}_\infty$ filtering problem for time varying systems.

Chapter 2: The $\mathbf{H}_\infty$ optimization problem is originally formulated in the frequency domain, which should be solved by the (J, J')-spectral factorization or Nevanlinna-Pick interpolation techniques. Therefore, Chapter 2 is first dedicated to providing a solution to the infinite-time horizon $\mathbf{H}_\infty$ filtering problem for time-invariant systems. As stated in the previous section, a complete parametrization of all $\mathbf{H}_\infty$ filters has not been derived for the infinite-horizon case in the previous works. Thus, we will derive a solvability condition and provide a complete parametrization of all solutions of the $\mathbf{H}_\infty$ filtering problem based on the model matching approach and (J, J')-spectral factorization. The resulting solution is given in terms of a positive semi-definite solution to a certain indefinite algebraic Riccati equation (ARE), which is called 'an $\mathbf{H}_\infty$ algebraic Riccati equation'. The structure of the $\mathbf{H}_\infty$ filtering problem is also shown by using the chain scattering representation. Furthermore, the $\mathbf{H}_\infty$ prediction problem is solved by making use of the results in the filtering problem.

Chapter 3: This chapter considers the $\mathbf{H}_\infty$ filtering problem with frequency constraints on the unit circle of the complex plane. If the system is subject to step or periodic disturbances, the state estimates may be degraded due to the biases or periodic fluctuations. In order to remove these undesirable effects, we impose boundary constraints such that the transfer functions from the step or periodic disturbances to the estimation error must be zero at certain frequency points on the unit circle. Based on the Nevanlinna-Pick interpolation technique, we develop a method for adjusting the free parameter of the $\mathbf{H}_\infty$ filter derived in the previous chapter so that the boundary constraints are satisfied. A numerical example is also given in order to demonstrate the applicability of the proposed design method.

Chapter 4: Since the state-space solution to the $\mathbf{H}_\infty$ filtering problem is expressed by the positive semi-definite stabilizing solution of the $\mathbf{H}_\infty$ ARE, the performance of the $\mathbf{H}_\infty$ filter depends on the stabilizing solution. Therefore, the analyses of the $\mathbf{H}_\infty$ARE are very important. In this chapter, we study some properties of the $\mathbf{H}_\infty$ ARE and the analysis of the $\mathbf{H}_\infty$ filter. We first derive the infimum of γ, for which a stabilizing solution to the $\mathbf{H}_\infty$ ARE exists, and show that the positive semi-definite stabilizing solution has the monotonicity and convexity properties with respect to γ.

Multi-objective filter design problems including $\mathbf{H}_2/\mathbf{H}_\infty$ filtering problem aim at achieving an additional design specification by using the free parameter contained in the $\mathbf{H}_\infty$ filter. Since the set of the free parameter is characterized by the stabilizing solution to the $\mathbf{H}_\infty$ ARE, we study the behavior of this set when γ changes based on the above properties of the $\mathbf{H}_\infty$ ARE. Such analyses of the $\mathbf{H}_\infty$ filter will provide a guideline for designing an $\mathbf{H}_\infty$ filter.

Chapter 5: This chapter considers the finite-time horizon minimax state estimation problems closely related to the $\mathbf{H}_\infty$ state estimation problems. As shown in Section 1.1, the $\mathbf{H}_\infty$ filtering problem has a certain minimax property. However, the frequency domain approach proposed in the previous chapters does not directly provide this property since it merely minimizes the largest singular value of a certain transfer matrix. In order to understand the minimax property of the $\mathbf{H}_\infty$ filtering problem, it is essential to exploit the game theoretic approach in the time domain. Based on the Lagrange multiplier technique, we show that the minimax state estimators are identical to the $\mathbf{H}_\infty$ estimators in both filtering and prediction cases. Furthermore, necessary and sufficient conditions for the existence of the minimax state estimators are given in terms of an $\mathbf{H}_\infty$-type Riccati difference equation (RDE) satisfying the positive definiteness of certain matrices.

Chapter 6: As shown in Section 1.3, a number of methods for solving the $\mathbf{H}_\infty$ filtering problems including mixed $\mathbf{H}_2/\mathbf{H}_\infty$ problems have been reported. However, the analysis of the estimation performance of the $\mathbf{H}_\infty$ filter has received much less attention. Thus, in this chapter, we will investigate the performance of the central $\mathbf{H}_\infty$ filter by using RDEs. First, by comparing the $\mathbf{H}_\infty$ and $\mathbf{H}_2$ (Kalman filtering) RDEs, we first consider the estimation performance in the case when the underlying disturbance is zero mean white noise. Next, we clarify the relationship between the prescribed $\mathbf{H}_\infty$ error bound and the performance of the central $\mathbf{H}_\infty$ filter based on the monotonicity of the $\mathbf{H}_\infty$ RDE. Also, for the convergence of the solution of the $\mathbf{H}_\infty$ RDE, we provide a sufficient condition, which connects the finite and infinite horizon $\mathbf{H}_\infty$ filtering problems. We also provide a solution of the $\mathbf{H}_\infty$ fixed-lag smoothing problem by reducing the problem to a usual $\mathbf{H}_\infty$ filtering problem.

Chapter 7: In this chapter, we will provide an alternative game theoretic interpretations of the central $\mathbf{H}_\infty$ filter and predictor. It may be noted that we have derived solutions to the minimax state estimation problems in the deterministic framework in Chapter 5. We will consider an alternative minimax state estimation problems in the stochastic setting, which are discrete-time equivalences to the problem discussed in [54]. It is shown that the $\mathbf{H}_\infty$ filter and predictor are generated by the minimizer's saddle-point policies to certain stochastic minimax filtering and prediction problems, respectively. Thus, the results of this chapter justify the application of the $\mathbf{H}_\infty$ state estimators to the stochastic systems.

Chapter 8: This chapter summarizes the results obtained in this thesis, and discuss the direction of the future research.

Chapter 2

A Model Matching Approach to $\mathbf{H}_\infty$ Filtering Problem

1. Introduction

This chapter considers the state-space solution to the $\mathbf{H}_\infty$ filtering problem for linear time-invariant systems. As shown in Chapter 1, the discrete-time $\mathbf{H}_\infty$ filtering problem has been considered from various points of view [52],[53],[55]. In these works, however, a complete parametrization of all discrete-time $\mathbf{H}_\infty$ filters was not given. Thus, in this chapter, we will derive a complete parametrization of all $\mathbf{H}_\infty$ filters based on the model matching approach. The model matching approach to robust state estimation was first formulated in [9] using a parametrization of stable unbiased filters, though a complete solution was not given. We first reduce the $\mathbf{H}_\infty$ filtering problem to a model matching problem (MMP) using the parametrization of all stable unbiased filters [14]. The MMP has been extensively studied by many researchers [11],[15],[28]. We give a state-space solution to the MMP based on the (J, J')-spectral factorization approach[15]. The present approach gives a straightforward proof in the pure frequency domain and a clear understanding of the structure of the $\mathbf{H}_\infty$ filtering problem even though the process disturbance and the measurement noise are correlated. It may be also noted that the results in this chapter are the discrete-time counterpart of those in [46].

Furthermore, the solution to the $\mathbf{H}_\infty$ prediction problem is given as a special case of the $\mathbf{H}_\infty$ filtering problem, whereas the problem was solved in a different setting from the $\mathbf{H}_\infty$ filtering problem in the previous works [52],[53]. The present approach provides a unified solution to the $\mathbf{H}_\infty$ filtering and prediction problems.

2. Problem Formulation

We consider a linear discrete-time system described by

$$x_{k+1} = Ax_k + Bd_k \tag{2.2.1}$$

$$y_k = Cx_k + Dd_k \tag{2.2.2}$$

where $x_k \in \mathbf{R}^n, y_k \in \mathbf{R}^q$ and $d_k \in \mathbf{R}^m$ are the state vector, the measurement and the unknown disturbance, respectively. We also assume that d_k is an arbitrary $\mathbf{L}_2$ signal. Let $z_k \in \mathbf{R}^p$ be the linear combination of the state variables given by

$$z_k = Lx_k, \quad L \neq 0 \tag{2.2.3}$$

The matrices A, B, C, D and L are constant matrices of appropriate dimensions.

The following standard conditions are assumed to hold.

(A1) (C, A) is detectable.

(A2) $\operatorname{rank} \begin{bmatrix} A - e^{j\omega}I_n & B \\ C & D \end{bmatrix} = n + q, \quad \forall \omega \in \mathbf{R}$

We wish to estimate z_k based on the measurement set $\{y_t |\ t \le k\}$ under the above assumptions. Let $\hat{z}_k$ be the estimate of z_k and $T_f(\sigma)$ be the transfer matrix of the filter, namely,

$$\hat{z} = T_f(\sigma)y \tag{2.2.4}$$

We also define the filtered estimation error by $e_k = z_k - \hat{z}_k$. Then, we see from (2.2.1)–(2.2.4) that

$$e = z - \hat{z} = \{T_{zd}(\sigma) - T_f(\sigma)T_{yd}(\sigma)\}d \tag{2.2.5}$$

where the transfer matrices from d_k to z_k and y_k are given by

$$T_{zd}(\sigma) = \left[\begin{array}{c|c} A & B \\ \hline L & 0 \end{array}\right], \quad T_{yd}(\sigma) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{2.2.6}$$

Hence the transfer matrix from d_k to e_k is given by

$$T_{ed}(\sigma) = T_{zd}(\sigma) - T_f(\sigma)T_{yd}(\sigma) \tag{2.2.7}$$

We consider the following design specifications.

(S1) $T_f(\sigma) \in \mathbf{RH}_\infty^{p\times q}$

(S2) $T_{ed}(\sigma) \in \mathbf{RH}_\infty^{p\times m}$

(S3) For a given scalar constant $\gamma > 0$,
(i) $\|T_{ed}\|_\infty < \gamma$, (ii) $\|T_{ed}\|_\infty \leq \gamma$

We also define the following sets of the $\mathbf{H}_\infty$ filters.

$\mathbf{A}(\gamma)$: the set of all $T_f(\sigma)$ satisfying (S1), (S2) and (S3-i)

$\bar{\mathbf{A}}(\gamma)$: the set of all $T_f(\sigma)$ satisfying (S1), (S2) and (S3-ii)

The $\mathbf{H}_\infty$ filtering problem is now stated as follows:

(a) Find a necessary and sufficient condition for $\mathbf{A}(\gamma) \neq \phi$.

(b) If $\mathbf{A}(\gamma)$ is not empty, parametrize all elements of $\mathbf{A}(\gamma)$ and $\bar{\mathbf{A}}(\gamma)$.

3. Preliminaries

In this section, we give some preliminary results on the (J, J')-spectral factorization, model matching problem and a (J, J')-lossless matrix. These results are useful for solving $\mathbf{H}_\infty$ filtering problem.

Given real symmetric matrices J, J' and a $p \times m$ transfer matrix $G(\sigma) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, the (J, J')-spectral factorization is the problem of finding a unimodular matrix $\Pi(\sigma)$ such that

$$G(\sigma)JG^{\sim}(\sigma) = \Pi(\sigma)J'\Pi^{\sim}(\sigma)$$

If such a matrix $\Pi(\sigma)$ exists, it is called a (J, J')-spectral factor. The following two lemmas are related to the state-space computation of (J, J')-spectral factorization.

Lemma 2.1: *Given real symmetric matrices $J \in \mathbf{R}^{m\times m}$, $J' \in \mathbf{R}^{p\times p}$ and a $p \times m$ transfer matrix $G(\sigma) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with A stable, there exists a unimodular matrix $\Pi(\sigma) \in \mathbf{GH}_\infty^{p\times p}$ such that*

$$G(\sigma)JG^{\sim}(\sigma) = \Pi(\sigma)J'\Pi^{\sim}(\sigma)$$

if and only if

(i) The following ARE has a unique stabilizing solution X.

$$X = AXA^{\mathrm{T}} - (AXC^{\mathrm{T}} + BJD^{\mathrm{T}})V^{-1}(AXC^{\mathrm{T}} + BJD^{\mathrm{T}})^{\mathrm{T}} + BJB^{\mathrm{T}} \tag{2.3.1}$$

where $V = DJD^{\mathrm{T}} + CXC^{\mathrm{T}}$.

(ii) There exists a nonsingular constant matrix $W \in \mathbf{R}^{p\times p}$ satisfying

$$WJ'W^{\mathrm{T}} = V \tag{2.3.2}$$

Then, such a transfer matrix $\Pi(\sigma)$ is given by

$$\Pi(\sigma) = \left[\begin{array}{c|c} A & K \\ \hline C & I_p \end{array}\right] W \tag{2.3.3}$$

$$K = (AXC^{\mathrm{T}} + BJD^{\mathrm{T}})V^{-1} \tag{2.3.4}$$

Proof: See Appendix 2.1. ■

Lemma 2.2: *For a given real symmetric matrix $V = \begin{bmatrix} V_{11} & V_{21}^{\mathrm{T}} \\ V_{21} & V_{22} \end{bmatrix} \in \mathbf{R}^{(q+p)\times(q+p)}$, we assume that $V_{11} > 0$ holds. Then a necessary and sufficient condition for the existence of a nonsingular matrix $W \in \mathbf{R}^{(q+p)\times(q+p)}$ satisfying $WJ_{qp}W^{\mathrm{T}} = V$ is that*

$$V_{21}V_{11}^{-1}V_{21}^{\mathrm{T}} - V_{22} > 0$$

Proof: See Appendix 2.2. ■

The following corollary is well known as the bounded real lemma.

Corollary 2.1: (Bounded Real Lemma)
For a given $p \times m$ transfer matrix $T(\sigma) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, suppose that (A, B) is stabilizable and (C, A) is detectable. Then, the following conditions are equivalent.

(i) The matrix A is stable and $\|T\|_\infty < \gamma$.

(ii) There exists a positive semi-definite stabilizing solution of the ARE

$$X = AXA^{\mathrm{T}} + (AXC^{\mathrm{T}} + BD^{\mathrm{T}})V^{-1}(AXC^{\mathrm{T}} + BD^{\mathrm{T}})^{\mathrm{T}} + BB^{\mathrm{T}} \tag{2.3.5}$$

with $V := \gamma^2 I_p - DD^{\mathrm{T}} - CXC^{\mathrm{T}} > 0$.

Proof: Although the lemma is proved in [6] and [52], we give another proof based on the (J, J')-spectral factorization. Assume that A is stable and $\|T\|_\infty < \gamma$. Then there exists a unimodular matrix $T_o(\sigma)$ satisfying $\gamma^2 I_p - TT^\sim = T_o T_o^\sim$. Thus, by taking $J = \begin{bmatrix} \gamma^2 I_p & 0 \\ 0 & -I_m \end{bmatrix}$, $J' = I_p$ and $G(\sigma) = [I_p \;\; T(\sigma)]$ in Lemma 2.1, we see that the ARE (2.3.5) has a stabilizing solution X with $V > 0$. Moreover, since A is stable, X is positive semi-definite by Lyapunov's theorem.

Conversely, if $X \geq 0$ is a stabilizing solution of the ARE (2.3.5) with $V > 0$, then A is stable by Lyapunov's theorem. Moreover, it follows from Lemma 2.1 that there exists a matrix $T_o(\sigma) \in \mathbf{GH}_\infty^{p\times p}$ satisfying $\gamma^2 I_p - TT^\sim = T_o T_o^\sim$. This implies that $\|T\|_\infty < \gamma$. ∎

The next lemma gives a connection between a model matching problem and the (J, J')-spectral factorization.

Lemma 2.3: *For given $T_1(\sigma) \in \mathbf{RL}_\infty^{p\times m}$ and $T_2(\sigma) \in \mathbf{RL}_\infty^{q\times m}$, suppose that $G(\sigma) := \begin{bmatrix} T_2 & 0 \\ T_1 & -I_p \end{bmatrix}$ has a right inverse in $\mathbf{RL}_\infty^{(m+p)\times(q+p)}$. Then the following are equivalent.*
(i) There exists a $Q(\sigma) \in \mathbf{RH}_\infty^{p\times q}$ satisfying $\|T_1 - QT_2\|_\infty < \gamma$.
(ii) There exists a unimodular matrix $\Pi(\sigma) = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \in \mathbf{GH}_\infty^{(q+p)\times(q+p)}$ such that

$$G(\sigma) J_{mp} G^\sim(\sigma) = \Pi(\sigma) J_{qp} \Pi^\sim(\sigma), \qquad \Pi_{11}(\sigma) \in \mathbf{GH}_\infty^{q\times q}$$

Proof: The proof is immediate from Theorem 2.4 of [15]. ∎

The notion of a (J, J')-lossless matrix is very important for deriving the parametrization of $\mathbf{H}_\infty$ filters.

Definition 2.1: *Given symmetric matrices $J \in \mathbf{R}^{m\times m}$, $J' \in \mathbf{R}^{p\times p}$, a transfer matrix $\Theta(\sigma) \in \mathbf{RL}_\infty^{p\times m}$ is called (J, J')-lossless if it satisfies*

$$\Theta(\sigma) J \Theta^\sim(\sigma) = J' \;\; \forall \sigma \in \mathbf{C}$$

$$\Theta(\sigma) J \Theta^{\mathrm{H}}(\sigma) \leq J' \;\; \forall \sigma \text{ s.t. } |\sigma| \geq 1$$

A remarkable property of a (J, J')-lossless matrix is shown in the following lemma.

Lemma 2.4: *Suppose that* $\Theta(\sigma) = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \in \mathbf{RL}_\infty^{(q+p)\times(m+p)}$ *is* (J_{mp}, J_{qp})*-lossless, and define*

$$\Phi(\sigma) = (U\Theta_{12} + \Theta_{22})^{-1}(U\Theta_{11} + \Theta_{21})$$

Then we have

(i) $\Phi(\sigma) \in \mathbf{BH}_\infty^{p\times m}$ *if and only if* $U(\sigma) \in \mathbf{BH}_\infty^{p\times q}$.

(ii) $\Phi(\sigma) \in \overline{\mathbf{BH}}_\infty^{p\times m}$ *if and only if* $U(\sigma) \in \overline{\mathbf{BH}}_\infty^{p\times q}$.

Proof: For the proof, see the reference [7]. ∎

4. Solution via (J, J')-Spectral Factorization

In this section, we will give a solution of the $\mathbf{H}_\infty$ filtering problem based on the (J, J')-spectral factorization approach. Since $T_{yd}(\sigma)$ and $T_{zd}(\sigma)$ may not be stable in general, we first need to characterize the class of all filters satisfying (S1) and (S2). This is the filtering equivalent of the class of internally stabilizing controllers [11].

Lemma 2.5: *The set of all filters satisfying (S1) and (S2) is given by*

$$T_{\mathfrak{f}}(\sigma) = T_{\mathfrak{f}1}(\sigma) - Q(\sigma)T_{\mathfrak{f}2}(\sigma) \tag{2.4.1}$$

$$T_{\mathfrak{f}1}(\sigma) = \left[\begin{array}{c|c} A_H & H \\ \hline L & 0 \end{array}\right], \quad T_{\mathfrak{f}2}(\sigma) = \left[\begin{array}{c|c} A_H & H \\ \hline C & -I_q \end{array}\right] \tag{2.4.2}$$

where $Q(\sigma)$ *is an arbitrary transfer matrix in* $\mathbf{RH}_\infty^{p\times q}$, *and where* $H \in \mathbf{R}^{n\times q}$ *is a matrix such that* $A_H := A - HC$ *is stable.*

Proof: See Appendix 2.3. ∎

The filter $T_{\mathfrak{f}}(\sigma)$ is strictly proper if and only if $Q(\sigma)$ is strictly proper. If $T_{\mathfrak{f}}(\sigma)$ is strictly proper, then it does not use the measurement y_k for the estimation at time k, namely $T_{\mathfrak{f}}(\sigma)$ is a predictor. Therefore, the above parametrization includes both filters and predictors.

We assume that $T_{\mathfrak{f}}(\sigma)$ is expressed by (2.4.1) and (2.4.2). Then, substituting (2.4.1) into (2.2.7) yields

$$T_{ed}(\sigma) = T_1(\sigma) - Q(\sigma)T_2(\sigma) \tag{2.4.3}$$

$$T_1(\sigma) = \left[\begin{array}{c|c} A_H & B_H \\ \hline L & 0 \end{array}\right], \quad T_2(\sigma) = \left[\begin{array}{c|c} A_H & B_H \\ \hline C & D \end{array}\right] \tag{2.4.4}$$

where $B_H = B - HD$. It thus remains to find a matrix $Q(\sigma) \in \mathbf{RH}_\infty^{p\times q}$ such that

$$\|T_1 - QT_2\|_\infty < \gamma \tag{2.4.5}$$

It may be noted that $T_{ed}(\sigma)$ is affine with respect to $Q(\sigma)$, and that $T_1(\sigma)$ and $T_2(\sigma)$ are stable. Thus, the $\mathbf{H}_\infty$ filtering problem reduces to a usual model matching problem (MMP) to which Lemma 2.3 is applicable.

Theorem 2.1: *The set $\mathbf{A}(\gamma)$ is non-empty if and only if*

(a) The algebraic Riccati equation

$$P = APA^{\mathrm{T}} - (AP\widehat{C}^{\mathrm{T}} + \widehat{S})V^{-1}(AP\widehat{C}^{\mathrm{T}} + \widehat{S})^{\mathrm{T}} + BB^{\mathrm{T}} \tag{2.4.6}$$

has a unique positive semi-definite stabilizing solution P, where

$$\begin{aligned} V = \begin{bmatrix} V_{11} & V_{21}^{\mathrm{T}} \\ V_{21} & V_{22} \end{bmatrix} &= \widehat{D}J_{mp}\widehat{D}^{\mathrm{T}} + \widehat{C}P\widehat{C}^{\mathrm{T}} \\ &= \begin{bmatrix} R + CPC^{\mathrm{T}} & CPL^{\mathrm{T}} \\ LPC^{\mathrm{T}} & -(\gamma^2 I_p - LPL^{\mathrm{T}}) \end{bmatrix} \end{aligned} \tag{2.4.7}$$

$$\widehat{C} = \begin{bmatrix} C \\ L \end{bmatrix}, \ \widehat{D} = \begin{bmatrix} D & 0 \\ 0 & -I_p \end{bmatrix}, \ \widehat{S} = \begin{bmatrix} BD^{\mathrm{T}} & 0 \end{bmatrix} \tag{2.4.8}$$

(b) For such a solution P, the following inequality holds.

$$\widehat{V} := V_{21}V_{11}^{-1}V_{21}^{\mathrm{T}} - V_{22} > 0 \tag{2.4.9}$$

Proof:

(i) Reduction to a (J_{mp}, J_{qp})- spectral factorization problem: We have only to consider the existence of a matrix $Q(\sigma) \in \mathbf{RH}_\infty^{p\times q}$ satisfying (2.4.5). We define

$$G(\sigma) = \begin{bmatrix} T_2(\sigma) & 0 \\ T_1(\sigma) & -I_p \end{bmatrix} = \left[\begin{array}{c|cc} A_H & B_H & 0 \\ \hline C & D & 0 \\ L & 0 & -I_p \end{array}\right] \tag{2.4.10}$$

where $\widehat{B}_H = \begin{bmatrix} B_H & 0 \end{bmatrix}$. Under the assumptions (A1) and (A2), $G(\sigma)$ has a right inverse in $\mathbf{RL}_\infty^{(m+p)\times(q+p)}$. Thus, we see from Lemma 2.3 that $\mathbf{A}(\gamma) \neq \phi$ holds if and only if there exists a unimodular matrix $\Pi(\sigma) = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \in \mathbf{GH}_\infty^{(q+p)\times(q+p)}$ satisfying

$$G(\sigma)J_{mp}G^{\sim}(\sigma) = \Pi(\sigma)J_{qp}\Pi^{\sim}(\sigma) \qquad (2.4.11)$$

with $\Pi_{11}(\sigma) \in \mathbf{GH}_\infty^{q\times q}$.

(ii) Derivation of the ARE (2.4.6): From Lemma 2.1, there exists a unimodular matrix $\Pi(\sigma)$ satisfying (2.4.11) if and only if there exists a unique stabilizing solution P of the ARE

$$\begin{aligned} P = {} & A_H P A_H^{\mathrm{T}} - (A_H P\widehat{C}^{\mathrm{T}} + \widehat{B}_H J_{mp}\widehat{D}^{\mathrm{T}}) \\ & \times V^{-1}(A_H P\widehat{C}^{\mathrm{T}} + \widehat{B}_H J_{mp}\widehat{D}^{\mathrm{T}})^{\mathrm{T}} + \widehat{B}_H J_{mp}\widehat{B}_H^{\mathrm{T}} \end{aligned} \qquad (2.4.12)$$

and there exists a nonsingular matrix $W \in \mathbf{R}^{(q+p)\times(q+p)}$ satisfying

$$WJ_{qp}W^{\mathrm{T}} = V \qquad (2.4.13)$$

It is obvious from (2.4.7) that

$$\begin{bmatrix} R + CPC^{\mathrm{T}} & CPL^{\mathrm{T}} \end{bmatrix} V^{-1} = \begin{bmatrix} I_q & 0 \end{bmatrix} \qquad (2.4.14)$$

Hence, we easily see that (2.4.12) is equivalent to (2.4.6). Furthermore, we define

$$K_H = (A_H P\widehat{C}^{\mathrm{T}} + \widehat{B}_H J_{mp}\widehat{D}^{\mathrm{T}})V^{-1} \qquad (2.4.15)$$

$$K = (AP\widehat{C}^{\mathrm{T}} + \widehat{S})V^{-1} \qquad (2.4.16)$$

$$A_K = A - K\widehat{C} \qquad (2.4.17)$$

Then, from (2.4.14), we obtain $K_H = K - H[I_q\ 0]$, and thus $A_K = A_H - K_H\widehat{C}$ holds. It follows that A_K is stable since P is a stabilizing solution of (2.4.12). This implies that P is also a stabilizing solution of the ARE (2.4.6).

(iii) Inequality (2.4.9) and the positive semi-definiteness of P: We hereafter assume that the ARE (2.4.6) has a unique stabilizing solution P.

If there exists a matrix $W \in \mathbf{R}^{(q+p)\times(q+p)}$ satisfying (2.4.13), then, from Lemma 2.3, the (J_{mp}, J_{qp})-spectral factor satisfying (2.4.11) is given by

$$\Pi(\sigma) = \left[\begin{array}{c|c} A_H & K_H \\ \hline \widehat{C} & I_{q+p} \end{array}\right] W = \left[\begin{array}{c|cc} A_H & K_H W_1 & K_H W_2 \\ \hline C & W_{11} & W_{12} \\ L & W_{21} & W_{22} \end{array}\right] \tag{2.4.18}$$

where $W \in \mathbf{R}^{(q+p)\times(q+p)}$ is appropriately partitioned as

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \quad W_1 = \begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix}, \quad W_2 = \begin{bmatrix} W_{12} \\ W_{22} \end{bmatrix}.$$

If $\Pi_{11}^{-1}(\sigma) \in \mathbf{RH}_\infty^{q\times q}$ holds, W_{11} is invertible and

$$\Pi_{11}^{-1}(\sigma) = \left[\begin{array}{c|c} A_\infty & -K_H W_1 \\ \hline W_{11}^{-1} C & I_q \end{array}\right] W_{11}^{-1} \tag{2.4.19}$$

where $A_\infty := A_H - K_H W_1 W_{11}^{-1} C$. Since this realization is stabilizable and detectable, A_∞ is stable. It is also easy to verify from (2.4.14) that $A_\infty = A - K W_1 W_{11}^{-1} C$. Moreover, (2.4.6) is expressed as

$$P = A_\infty P A_\infty^{\mathrm{T}} + N_\infty N_\infty^{\mathrm{T}} \tag{2.4.20}$$

where

$$N_\infty = \begin{bmatrix} B - K W_1 W_{11}^{-1} D & \gamma K (W_2 - W_1 W_{11}^{-1} W_{12}) \end{bmatrix}$$

Since the second term in the right hand side of (2.4.20) is positive semi-definite and since A_∞ is stable, $P \geq 0$ holds. Also, since $G(\sigma)$ is invertible in $\mathbf{RL}_\infty$, V must be nonsingular. Together with $P \geq 0$, this implies $V_{11} = R + CPC^{\mathrm{T}} > 0$. It thus follows from Lemma 2.2 that $\widehat{V} = V_{21} V_{11}^{-1} V_{21}^{\mathrm{T}} - V_{22} > 0$.

Conversely, we assume that the conditions (a),(b) hold. Then, from (2.4.7), we get $V_{11} = R + CPC^{\mathrm{T}} > 0$. Hence, from Lemma 2.2, there exists a nonsingular $W \in \mathbf{R}^{(q+p)\times(q+p)}$ satisfying (2.4.13). Moreover, there exists a unimodular matrix $\Pi(\sigma) \in \mathbf{RH}_\infty^{(q+p)\times(q+p)}$ satisfying (2.4.11) and it is given by (2.4.18). Also, since $W_{11} W_{11}^{\mathrm{T}} = V_{11} + \gamma^2 W_{12} W_{12}^{\mathrm{T}} > 0$ holds from $V_{11} > 0$ and (2.4.7), W_{11} is invertible. Thus, we can express (2.4.6) as (2.4.20). Since A_K is stable, (A_∞, K) is stabilizable. This implies that the pair (A_∞, N_∞) is also stabilizable [49]. It thus follows from Lyapunov's theorem that A_∞ is stable, so $\Pi_{11}^{-1}(\sigma) \in \mathbf{RH}_\infty^{q\times q}$ holds. Therefore, from Lemma 2.3, the MMP (2.4.5) is solvable. ■

Next we give a parametrization of all $\mathbf{H}_\infty$ filters $T_f(\sigma)$.

Theorem 2.2: *Suppose that the set $\mathbf{A}(\gamma)$ is not empty. Then the parametrization of all $T_f(\sigma) \in \mathbf{A}(\gamma)$ (resp. $T_f(\sigma) \in \tilde{\mathbf{A}}(\gamma)$) is given by*

$$T_f(\sigma) = -(U\Omega_{12} - \Omega_{22})^{-1}(U\Omega_{11} - \Omega_{21}) \tag{2.4.21}$$

$$\Omega(\sigma) = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} = W^{-1} \left[\begin{array}{c|cc} A_K & -K_1 & -K_2 \\ \hline C & I_q & 0 \\ L & 0 & I_p \end{array}\right] \tag{2.4.22}$$

where $U(\sigma)$ is an arbitrary transfer matrix in $\mathbf{BH}_\infty^{p\times q}$ (resp. $\overline{\mathbf{B}}\mathbf{H}_\infty^{p\times q}$), and where $K = [K_1 \ \ K_2] \in \mathbf{R}^{n\times(q+p)}$, A_K and W are defined by (2.4.16), (2.4.17) and (2.4.13), respectively.

Proof: We define

$$\begin{aligned}\Theta(\sigma) = \begin{bmatrix} \Theta_{11}(\sigma) & \Theta_{12}(\sigma) \\ \Theta_{21}(\sigma) & \Theta_{22}(\sigma) \end{bmatrix} &= \Pi^{-1}(\sigma)G(\sigma) \\ &= W^{-1} \left[\begin{array}{c|cc} A_K & B - K_1 D & K_2 \\ \hline C & D & 0 \\ L & 0 & -I_p \end{array}\right] \end{aligned} \tag{2.4.23}$$

Since $G(\sigma)J_{mp}G^{\sim}(\sigma) = \Pi(\sigma)J_{qp}\Pi^{\sim}(\sigma)$ holds, we get

$$\Theta(\sigma)J_{mp}\Theta^{\sim}(\sigma) = J_{qp}$$

Some simple calculations yield

$$\begin{aligned} &J_{qp} - \Theta(\sigma)J_{mp}\Theta^{\mathrm{H}}(\sigma) \\ &\quad = (|\sigma|^2 - 1) \left[\begin{array}{c|c} A_K & I_n \\ \hline W^{-1}\widehat{C} & 0 \end{array}\right] P \left[\begin{array}{c|c} A_K & I_n \\ \hline W^{-1}\widehat{C} & 0 \end{array}\right]^{\mathrm{H}} \geq 0 \quad \forall \sigma \text{ s.t. } |\sigma| \geq 1 \end{aligned}$$

Thus $G(\sigma)$ has the (J_{mp}, J_{qp})-lossless factorization $G(\sigma) = \Pi(\sigma)\Theta(\sigma)$ with a (J_{mp}, J_{qp})-lossless matrix $\Theta(\sigma)$ and a unimodular matrix $\Pi(\sigma)$. Let $X_1(\sigma) \in \mathbf{RH}_\infty^{q\times p}$ and $X_2(\sigma) \in \mathbf{RH}_\infty^{p\times p}$ be defined by

$$\begin{bmatrix} X_1(\sigma) & X_2(\sigma) \end{bmatrix} = \begin{bmatrix} -Q(\sigma) & I_p \end{bmatrix} \Pi(\sigma) \tag{2.4.24}$$

Then we get

$$\begin{aligned}\left[\begin{array}{cc} T_{ed}(\sigma) & -I_p \end{array}\right] &= \left[\begin{array}{cc} -Q(\sigma) & I_p \end{array}\right] G(\sigma) \\ &= \left[\begin{array}{cc} -Q(\sigma) & I_p \end{array}\right] \Pi(\sigma)\Theta(\sigma) \\ &= \left[\begin{array}{cc} X_1(\sigma) & X_2(\sigma) \end{array}\right] \Theta(\sigma) \end{aligned} \tag{2.4.25}$$

Hence $T_{ed}(\sigma)$ is expressed as

$$\begin{aligned} T_{ed}(\sigma) &= -\{(-U)\Theta_{12} + \Theta_{22}\}^{-1}\{(-U)\Theta_{11} + \Theta_{21}\} \\ &= -(U\Theta_{12} - \Theta_{22})^{-1}(U\Theta_{11} - \Theta_{21}) \end{aligned} \tag{2.4.26}$$

where $U(\sigma) := -X_2^{-1}(\sigma)X_1(\sigma)$. Since $\Theta(\sigma)$ is (J_{mp}, J_{qp})-lossless, it follows from Lemma 2.4 that $T_{ed}(\sigma) \in \mathbf{BH}_\infty^{p\times m}$ if and only if $U(\sigma) \in \mathbf{BH}_\infty^{q\times p}$ (resp. $T_{ed}(\sigma) \in \overline{\mathbf{B}}\mathbf{H}_\infty^{p\times m}$ iff $U(\sigma) \in \overline{\mathbf{B}}\mathbf{H}_\infty^{q\times p}$). Furthermore, from (2.4.24), we see that

$$-Q(\sigma) = X_1\hat{\Pi}_{11} + X_2\hat{\Pi}_{21} \tag{2.4.27}$$

$$I_p = X_1\hat{\Pi}_{12} + X_2\hat{\Pi}_{22} \tag{2.4.28}$$

where $\Pi^{-1}(\sigma)$ is partitioned as $\Pi^{-1}(\sigma) = \begin{bmatrix} \hat{\Pi}_{11} & \hat{\Pi}_{12} \\ \hat{\Pi}_{21} & \hat{\Pi}_{22} \end{bmatrix}$. Hence we get

$$Q(\sigma) = -(U\hat{\Pi}_{12} - \hat{\Pi}_{22})^{-1}(U\hat{\Pi}_{11} - \hat{\Pi}_{21}) \tag{2.4.29}$$

Substituting this into (2.4.1) yields

$$T_f(\sigma) = -(U\hat{\Pi}_{12} - \hat{\Pi}_{22})^{-1}\left\{U\left(\hat{\Pi}_{12}T_{f1} + \hat{\Pi}_{11}T_{f2}\right) - \left(\hat{\Pi}_{22}T_{f1} + \hat{\Pi}_{21}T_{f2}\right)\right\}$$

Therefore, we obtain the parametrization of (2.4.21)–(2.4.22) by defining $\Omega(\sigma)$ as

$$\begin{aligned}\Omega(\sigma) = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} &= \begin{bmatrix} \hat{\Pi}_{11} & \hat{\Pi}_{12} \\ \hat{\Pi}_{21} & \hat{\Pi}_{22} \end{bmatrix} \begin{bmatrix} -T_{f2} & 0 \\ -T_{f1} & I_p \end{bmatrix} \\ &= W^{-1} \left[\begin{array}{c|cc} A_K & -K_1 & -K_2 \\ \hline C & I_q & 0 \\ L & 0 & I_p \end{array}\right] \end{aligned}$$

■

Remark 2.1: It may be noted from Theorems 2.1 and 2.2 that the solution of the $\mathbf{H}_\infty$ filtering problem is independent of the constant matrix H which is introduced in Lemma 2.5.

Remark 2.2: It is easily seen from the proof of Lemma 2.2 that if the $\mathbf{H}_\infty$ filtering problem is solvable, we can take $W_{12} = 0$ as in (A.2.26) without loss of generality. Then, $\Omega(\sigma)$ of Theorem 2.2 is given by

$$\Omega(\sigma) = \left[\begin{array}{c|cc} A_K & -K_1 & -K_2 \\ \hline W_{11}^{-1}C & W_{11}^{-1} & 0 \\ L' & -W_{22}^{-1}W_{21}W_{11}^{-1} & W_{22}^{-1} \end{array}\right] \tag{2.4.30}$$

where $L' = W_{22}^{-1}(L - W_{21}W_{11}^{-1}C)$. In this case, by taking $U(\sigma) = 0$, we obtain

$$T_f(\sigma) = -\Omega_{22}^{-1}(\sigma)\Omega_{21}(\sigma) = \left[\begin{array}{c|c} A - K_\infty C & K_\infty \\ \hline L - M_\infty C & M_\infty \end{array}\right] \tag{2.4.31}$$

It is easy to verify that

$$K_\infty = KW_1W_{11}^{-1} = (APC^{\mathrm{T}} + BD^{\mathrm{T}})(R + CPC^{\mathrm{T}})^{-1} \tag{2.4.32}$$

$$M_\infty = W_{21}W_{11}^{-1} = LPC^{\mathrm{T}}(R + CPC^{\mathrm{T}})^{-1} \tag{2.4.33}$$

Hereafter, we refer to this $\mathbf{H}_\infty$ filter as the central $\mathbf{H}_\infty$ filter or the central solution of the $\mathbf{H}_\infty$ filtering problem. It may be noted that, when γ tends to infinity, the $\mathbf{H}_\infty$ ARE (2.4.6) reduces to the Kalman filtering type ($\mathbf{H}_2$-type) ARE, and hence the central $\mathbf{H}_\infty$ filter reduces to the Kalman ($\mathbf{H}_2$-optimal) filter.

5. Structure of $\mathbf{H}_\infty$ Filtering Problem

In this section, we will study the structure of the $\mathbf{H}_\infty$ filtering problem using the chain scattering representation[29].

We consider a system described as

$$\begin{bmatrix} u_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ y_1 \end{bmatrix} \tag{2.5.1}$$

where (u_1, y_2) and (u_2, y_1) are the inputs and outputs of the system, respectively. The transfer matrix $\Sigma(\sigma) = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$ is called "a chain scattering matrix". This system can be illustrated as in Fig. 2.1.

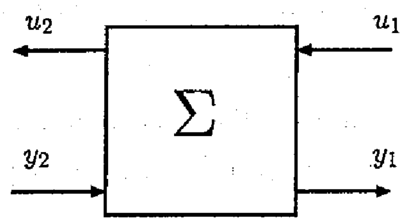


Fig. 2.1: Chain scattering representation

Using the chain scattering matrices $\Sigma_1(\sigma)$ and $\Sigma_2(\sigma)$, the input-output relationship of the cascade connection of two systems in Fig. 2.2 (a) is given by

$$\begin{bmatrix} u_3 \\ y_3 \end{bmatrix} = \Sigma_2(\sigma) \begin{bmatrix} u_2 \\ y_2 \end{bmatrix} = \Sigma_2(\sigma)\Sigma_1(\sigma) \begin{bmatrix} u_1 \\ y_1 \end{bmatrix} \tag{2.5.2}$$

This implies that the cascade connection of systems can be represented by using the product of the chain scattering matrices of each system (Fig. 2.2 (b)).

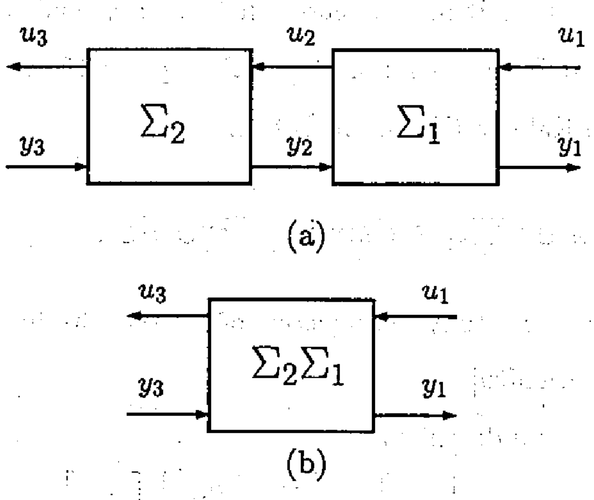


Fig. 2.2: Cascade connection of chain scattering matrices

We next consider a closed-loop system shown in Fig. 2.3. The input-output relationship of this system is described by

$$\Phi(\sigma) = -(Q\Sigma_{11} - \Sigma_{21})^{-1}(Q\Sigma_{12} - \Sigma_{22}) \tag{2.5.3}$$

As shown in Lemma 2.4, if $\Sigma(\sigma) \in \mathbf{RL}_{\infty}^{(q+p)\times(m+p)}$ is (J_{mp}, J_{qp})-lossless, then a necessary and sufficient condition for $\Phi(\sigma) \in \mathbf{BH}_{\infty}^{p\times m}$ is $Q(\sigma) \in \mathbf{BH}_{\infty}^{p\times q}$.

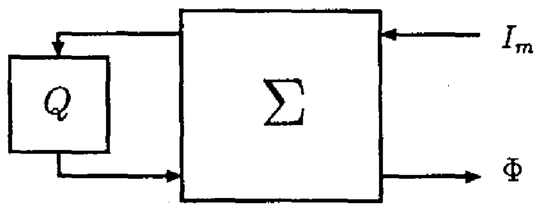


Fig. 2.3: Chain scattering representation of a closed-loop system

Based on the above properties of the chain scattering representation, the structure of the $\mathbf{H}_\infty$ filtering problem is illustrated in Fig. 2.4, where $\Sigma_{\mathsf{sys}}(\sigma)$ and $\Sigma_{\mathsf{f}}(\sigma)$ are the chain scattering matrices associated with the system (2.2.1)–(2.2.3) and the filter (2.4.1),(2.4.2), respectively.

$$\Sigma_{\mathsf{sys}}(\sigma) = \begin{bmatrix} T_{yd}(\sigma) & 0 \\ T_{zd}(\sigma) & -I_p \end{bmatrix}, \quad \Sigma_{\mathsf{f}}(\sigma) = \begin{bmatrix} -T_{\mathsf{f}2}(\sigma) & 0 \\ -T_{\mathsf{f}1}(\sigma) & I_p \end{bmatrix}$$

Note that $\Omega(\sigma)$ is independent of the matrix H in Lemma 2.5 because of the pole-zero cancellation between $\Pi^{-1}(\sigma)$ and $\Sigma_{\mathsf{f}}(\sigma)$. If the $\mathbf{H}_\infty$ filtering problem is solvable, then $\Sigma_{\mathsf{sys}}(\sigma)$ has a (J_{mp}, J_{qp})-lossless coprime factorization $\Sigma_{\mathsf{sys}} = \Omega^{-1}\Theta$ with $\Omega(\sigma)$ and $\Theta(\sigma)$ defined by (2.4.22) and (2.4.23). Conversely, suppose that $\Sigma_{\mathsf{sys}}(\sigma)$ has a (J_{mp}, J_{qp})-lossless coprime factorization $\Sigma_{\mathsf{sys}} = \Omega^{-1}\Theta$ without assuming any particular realizations of $\Omega(\sigma)$ and $\Theta(\sigma)$. Then, from Fig. 2.4) and Lemma 2.4, an $\mathbf{H}_\infty$ filter in $\mathbf{A}(\gamma)$ is given by (2.4.21) with $U(\sigma) \in \mathbf{BH}_\infty$. In summary, we have the following theorem.

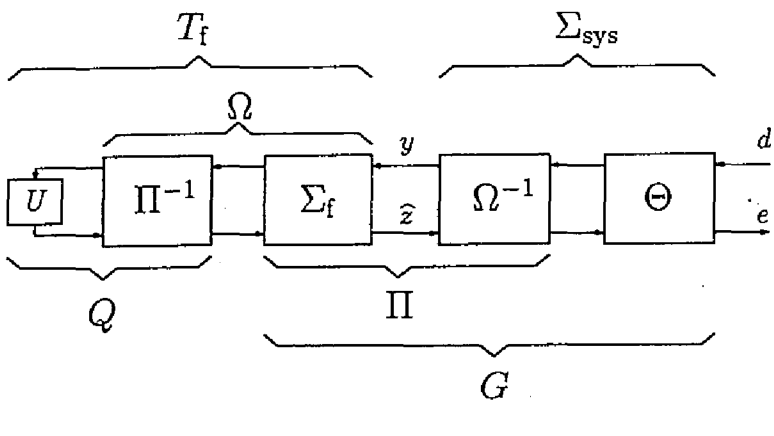


Fig. 2.4: Structure of $\mathbf{H}_\infty$ filtering problem

Theorem 2.3: *The set* $\mathbf{A}(\gamma)$ *is non-empty if and only if* $\Sigma_{sys}(\sigma)$ *has a* (J_{mp}, J_{qp})*-lossless coprime factorization of* $\Sigma_{sys} = \Omega^{-1}\Theta$, *where* $\Theta(\sigma) \in \mathbf{RH}_\infty^{(q+p)\times(m+p)}$ *is* (J_{mp}, J_{qp})*-lossless and* $\Omega(\sigma) \in \mathbf{RH}_\infty^{(q+p)\times(q+p)}$.

6. Solution of $\mathbf{H}_\infty$ Prediction Problem

From the discussion in Section 4, we see that the key properties of $\mathbf{H}_\infty$ filters lie in the biproper $\mathbf{H}_\infty$ filter of (2.4.31)–(2.4.33) which utilizes $\{y_t |\ t \le k\}$ for the estimation at time k. Therefore, even if the $\mathbf{H}_\infty$ filtering problem is solvable, there may not exist an $\mathbf{H}_\infty$ predictor (strictly proper $T_f(\sigma)$) which uses $\{y_t |\ t \le k-1\}$ rather than $\{y_t |\ t \le k\}$. In this section, we consider the $\mathbf{H}_\infty$ prediction problem as a special case of the $\mathbf{H}_\infty$ filtering problem.

Theorem 2.4: *Suppose that* $\mathbf{A}(\gamma) \ne \phi$ *holds. Then, a necessary and sufficient condition for the existence of an* $\mathbf{H}_\infty$ *predictor satisfying (S1)–(S3) is that* $V_{22} < 0$ *holds for the positive semi-definite stabilizing solution of the ARE (2.4.6).*

Proof: *Necessity:* Suppose that there exists an $\mathbf{H}_\infty$ predictor $T_f(\sigma)$ satisfying (S1)–(S3). By (2.4.21), $U\Omega_{11} - \Omega_{21}$ must be strictly proper for such a $T_f(\sigma)$. We now assume $W_{12} = 0$ without loss of generality, so that $\Omega(\sigma)$ is given by (2.4.30). Moreover, let the realization of $U(\sigma)$ be given by $U(\sigma) = \left[\begin{array}{c|c} A_U & B_U \\ \hline C_U & D_U \end{array}\right]$, so that

$$U(\sigma)\Omega_{11}(\sigma) - \Omega_{21}(\sigma) = \left[\begin{array}{cc|c} A_U & B_U W_{11}^{-1} C & B_U W_{11}^{-1} \\ 0 & A_K & -K_1 \\ \hline C_U & D_U W_{11}^{-1} C - L' & (D_U + W_{22}^{-1} W_{21}) W_{11}^{-1} \end{array}\right]$$

Thus we obtain $D_U = -W_{22}^{-1} W_{21}$.

Since $\|U\|_\infty < \gamma$ holds from Theorem 2.2, we get

$$\gamma^2 I_p - D_U D_U^{\mathrm{T}} = \gamma^2 I_p - W_{22}^{-1} W_{21} W_{21}^{\mathrm{T}} W_{22}^{-T} = -W_{22}^{-1} V_{22} W_{22}^{-T} > 0$$

Therefore, $V_{22} < 0$ holds.

Sufficiency : Suppose that $V_{22} < 0$ holds. Then,

$$W = \begin{bmatrix} (V_{11} - V_{21}^{\mathrm{T}} V_{22}^{-1} V_{21})^{1/2} & -\gamma^{-1} V_{21}^{\mathrm{T}} (-V_{22})^{-1/2} \\ 0 & \gamma^{-1} (-V_{22})^{1/2} \end{bmatrix} \tag{2.6.1}$$

is nonsingular and satisfies $W J_{qp} W^{\mathrm{T}} = V$. By taking W as in (2.6.1), we get the following parametrization from Theorem 2.2.

$$T_{\mathrm{f}}(\sigma) = -(U\Omega_{12} - \Omega_{22})^{-1}(U\Omega_{11} - \Omega_{21}) \tag{2.6.2}$$

$$\Omega(\sigma) = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A_K & -K_1 & -K_2 \\ \hline C' & W_{11}^{-1} & -W_{11}^{-1} W_{12} W_{22}^{-1} \\ W_{22}^{-1} L & 0 & W_{22}^{-1} \end{array}\right] \tag{2.6.3}$$

where $U(\sigma)$ is an arbitrary transfer matrix in $\mathbf{BH}_\infty^{p\times q}$ and $C' = W_{11}^{-1} C - W_{11}^{-1} W_{12} W_{22}^{-1} L$. Moreover, taking $U(\sigma) = 0$ yields an $\mathbf{H}_\infty$ predictor

$$T_{\mathrm{f}}(\sigma) = -\Omega_{22}^{-1}(\sigma)\Omega_{21}(\sigma) = \left[\begin{array}{c|c} A - K_1 C & K_1 \\ \hline L & 0 \end{array}\right] \tag{2.6.4}$$

This completes the proof. ∎

Remark 2.3: It is easy to verify that

$$K_1 = (A\bar{P}C^{\mathrm{T}} + BD^{\mathrm{T}})(R + C\bar{P}C^{\mathrm{T}})^{-1}$$

$$\bar{P} = P(I_n - \gamma^{-2} L^{\mathrm{T}} L P)^{-1}$$

Thus, as γ tends to infinity, the $\mathbf{H}_\infty$ predictor (2.6.4) reduces to Kalman predictor.

Remark 2.4: Since $V_{11} > 0$ holds, we get

$$\widehat{V} = V_{21}V_{11}^{-1}V_{21}^{\mathrm{T}} - V_{22} \geq -V_{22}$$

Thus we see from Theorems 2.1 and 2.4 that the existence condition of an $\mathbf{H}_\infty$ filter is more relaxed than that of an $\mathbf{H}_\infty$ predictor. This implies that there exists an $\mathbf{H}_\infty$ filter which achieves the smaller $\mathbf{H}_\infty$ error bound than any $\mathbf{H}_\infty$ predictors.

7. Concluding Remarks

In this chapter, we have given a solvability condition of the $\mathbf{H}_\infty$ filtering problem based on the model matching approach using (J, J')-spectral factorization. We have also derived a complete parametrization of all solutions. Similarly to the $\mathbf{H}_\infty$ control case, the free parameter of the parametrization can be used for achieving an additional design specification, e.g. $\mathbf{H}_2$ performance, as well as the $\mathbf{H}_\infty$ error bound. Such multi-objective design of an $\mathbf{H}_\infty$ filter will be discussed in the following chapters.

Furthermore, we have given a solution to the $\mathbf{H}_\infty$ prediction problem as a special case of the $\mathbf{H}_\infty$ filtering problem. The present approach provides a unified solution to the $\mathbf{H}_\infty$ filtering and prediction problems.

Appendix 2.1: Proof of Lemma 2.1

It may be noted that the continuous-time result is given in [15] and that the proof of the discrete-time result is given by using the bilinear transformation [16]. We here prove without using the bilinear transformation.

Sufficiency : We assume that there exist a stabilizing solution X to the ARE (2.3.1) and that a nonsingular matrix W satisfying (2.3.2) exists. We define $\Psi(\sigma) := G(\sigma)JG^{\sim}(\sigma)$ and let $\Pi(\sigma)$ and K be defined by (2.3.3) and (2.3.4), namely

$$\Pi(\sigma) = \left[\begin{array}{c|c} A & K \\ \hline C & I_p \end{array}\right] W$$

$$K = (AXC^{\mathrm{T}} + BJD^{\mathrm{T}})V^{-1}$$

Since X is a stabilizing solution of the ARE (2.3.1), it is straightforward to show that

$$\Pi^{-1}(\sigma) = W^{-1}\left[\begin{array}{c|c} A - KC & -K \\ \hline C & I_p \end{array}\right] \in \mathbf{RH}_\infty^{p\times p}$$

It remains to show that $\Psi(\sigma) = \Pi(\sigma)J'\Pi^{\sim}(\sigma)$.

From (2.3.1), we get

$$\begin{aligned} BJB^{\mathrm{T}} &= \sigma X\sigma^{-1} - AXA^{\mathrm{T}} + (AXC^{\mathrm{T}} + BJD^{\mathrm{T}})V^{-1}(AXC^{\mathrm{T}} + BJD^{\mathrm{T}})^{\mathrm{T}} \\ &= (\sigma I_n - A)X(\sigma^{-1}I_n - A^{\mathrm{T}}) + KVK^{\mathrm{T}} \\ &\quad +(\sigma I_n - A)XA^{\mathrm{T}} + AX(\sigma^{-1}I_n - A^{\mathrm{T}}) \end{aligned} \tag{A.2.1}$$

Pre-multiplying by $\Phi(\sigma) := C(\sigma I_n - A)^{-1}$ and post-multiplying by $\Phi^{\sim}(\sigma)$ yield

$$\begin{aligned} \Phi(\sigma)BJB^{\mathrm{T}}\Phi(\sigma)^{\sim} &= CXC^{\mathrm{T}} + \Phi(\sigma)KVK^{\mathrm{T}}\Phi^{\sim}(\sigma) \\ &\quad +\Phi(\sigma)AXC^{\mathrm{T}} + CXA^{\mathrm{T}}\Phi^{\sim}(\sigma) \end{aligned} \tag{A.2.2}$$

Since $G(\sigma) = \Phi(\sigma)B + D$, it follows from (A.2.2) that

$$\begin{aligned} \Psi(\sigma) &= DJD^{\mathrm{T}} + \Phi(\sigma)BJB^{\mathrm{T}}\Phi^{\sim}(\sigma) \\ &\quad +DJB^{\mathrm{T}}\Phi^{\sim}(\sigma) + \Phi(\sigma)BJD^{\mathrm{T}} \\ &= \left[\begin{array}{c|c} A & K \\ \hline C & I_p \end{array}\right] V \left[\begin{array}{c|c} A & K \\ \hline C & I_p \end{array}\right]^{\sim} \end{aligned} \tag{A.2.3}$$

Substituting (2.3.2) into (A.2.3) yields $\Psi(\sigma) = \Pi(\sigma)J'\Pi^{\sim}(\sigma)$. This completes the proof of sufficiency.

Necessity : We first consider the case where (C, A) is observable. The basic idea of the proof is due to [34]. Since A is stable, there exists a unique solution to the Lyapunov equation

$$X_1 = AX_1A^{\mathrm{T}} + BJB^{\mathrm{T}} \tag{A.2.4}$$

Then we get

$$BJB^{\mathrm{T}} = (\sigma I_n - A)X_1(\sigma^{-1}I_n - A^{\mathrm{T}}) + AX_1(\sigma^{-1}I_n - A^{\mathrm{T}}) + (\sigma I_n - A)XA^{\mathrm{T}}$$

Pre-multiplying by $\Phi(\sigma)$ and post-multiplying by $\Phi^{\sim}(\sigma)$ yield

$$\Phi(\sigma)BJB^{\mathrm{T}}\Phi^{\sim}(\sigma) = CX_1C^{\mathrm{T}} + \Phi(\sigma)AX_1C^{\mathrm{T}} + CX_1A^{\mathrm{T}}\Phi^{\sim}(\sigma)$$

Hence, we get

$$\begin{aligned}\Psi(\sigma) &= DJD^{\mathrm{T}} + \Phi(\sigma)BJB^{\mathrm{T}}\Phi^{\sim}(\sigma)\\ &\quad+\Phi(\sigma)BJD^{\mathrm{T}} + DJB^{\mathrm{T}}\Phi^{\sim}(\sigma)\\ &= DJD^{\mathrm{T}} + CX_1C^{\mathrm{T}} + \breve{G}(\sigma) + \breve{G}^{\sim}(\sigma)\end{aligned} \tag{A.2.5}$$

where

$$\breve{G}(\sigma) = \left[\begin{array}{c|c} A & AX_1C^{\mathrm{T}} + BJD^{\mathrm{T}} \\ \hline C & 0 \end{array}\right] \tag{A.2.6}$$

We now assume that there exists a unimodular matrix $\Pi(\sigma)$ such that $\Psi = \Pi J' \Pi^{\sim}$. From (A.2.5), we easily see that if λ is a pole of $\Psi(\sigma)$ then $1/\lambda$ is also a pole of $\Psi(\sigma)$, and that if $|\lambda| < 1$ then λ is an eigenvalue of A. Thus, we can take $\Pi(\sigma)$ as

$$\Pi(\sigma) = \left[\begin{array}{c|c} A & \Gamma \\ \hline C & W \end{array}\right], \quad W\text{: nonsingular}$$

Note that since (C, A) is observable, Γ is unique for C, A and $\Pi(\sigma)$. Since W is nonsingular, $V := WJ'W^{\mathrm{T}}$ is also nonsingular. Similarly to the derivation of (A.2.5) and (A.2.6), we obtain

$$\Psi(\sigma) = \Pi(\sigma)J'\Pi^{\sim}(\sigma) = V + CX_2C^{\mathrm{T}} + \breve{\Pi}(\sigma) + \breve{\Pi}^{\sim}(\sigma) \tag{A.2.7}$$

$$\breve{\Pi}(\sigma) = \left[\begin{array}{c|c} A & AX_2C^{\mathrm{T}} + \Gamma J'W^{\mathrm{T}} \\ \hline C & 0 \end{array}\right] \tag{A.2.8}$$

where X_2 is a unique solution to the Lyapunov equation

$$X_2 = AX_2A^{\mathrm{T}} + \Gamma J'\Gamma^{\mathrm{T}} \tag{A.2.9}$$

Since both $G'(\sigma)$ and $\Pi'(\sigma)$ are in $\mathbf{RH}_{\infty}^{p\times p}$ and strictly proper, comparing (A.2.5) with (A.2.7) yields

$$\breve{G}(\sigma) = \breve{\Pi}(\sigma) \tag{A.2.10}$$

$$DJD^{\mathrm{T}} + CX_1C^{\mathrm{T}} = V + CX_2C^{\mathrm{T}} \tag{A.2.11}$$

Moreover, since (C, A) is observable, from (A.2.6),(A.2.8) and (A.2.10), we get

$$AX_1C^{\mathrm{T}} + BJD^{\mathrm{T}} = AX_2C^{\mathrm{T}} + \Gamma J'W^{\mathrm{T}} \tag{A.2.12}$$

Define $X := X_1 - X_2$. It then follows from (A.2.11),(A.2.12) and the invertibility of V that

$$V = DJD^{\mathrm{T}} + CXC^{\mathrm{T}} \tag{A.2.13}$$

$$\Gamma = KW \tag{A.2.14}$$

$$K = (AXC^{\mathrm{T}} + BJD^{\mathrm{T}})V^{-1} \tag{A.2.15}$$

Substituting above equations into (A.2.9) yields

$$X_2 = AX_2A^{\mathrm{T}} + (AXC^{\mathrm{T}} + BJD^{\mathrm{T}})V^{-1}(CXA^{\mathrm{T}} + DJB^{\mathrm{T}})$$

By subtracting this from (A.2.4), we obtain the ARE (2.3.1). Note that X is unique since X_1, X_2 and Γ are unique. Moreover, Π^{-1} is given by

$$\Pi^{-1}(\sigma) = \left[\begin{array}{c|c} A - \Gamma W^{-1}C & -W^{-1}\Gamma \\ \hline W^{-1}C & W^{-1} \end{array}\right] = \left[\begin{array}{c|c} A - KC & -K \\ \hline W^{-1}C & W^{-1} \end{array}\right]$$

Since $\Pi(\sigma)$ is unimodular, $A - KC$ is stable, i.e. X is a stabilizing solution.

Next, we consider the case where (C, A) is not observable. It suffices to show that there exists a stabilizing solution of the ARE (2.3.1). Without loss of generality, we assume that A, B and C are in the canonical form

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{A.2.16}$$

where (C_1, A_{11}) is observable and A_{22} is stable. Also, it is easy to verify that $G(\sigma) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$. According to (A.2.16), we partition X as $X = \begin{bmatrix} X_{11} & X_{21}^{\mathrm{T}} \\ X_{21} & X_{22} \end{bmatrix}$. Then the ARE (2.3.1) reduces to the following simultaneous equations.

$$V = DJD^{\mathrm{T}} + C_1 X_{11} C_1^{\mathrm{T}} \tag{A.2.17}$$

$$\begin{aligned} X_{11} &= A_{11}X_{11}A_{11}^{\mathrm{T}} + B_1 J B_1^{\mathrm{T}} - (A_{11}X_{11}C_1^{\mathrm{T}} + B_1 J D^{\mathrm{T}}) \\ &\quad \times V^{-1}(A_{11}X_{11}C_1^{\mathrm{T}} + B_1 J D^{\mathrm{T}})^{\mathrm{T}} \end{aligned} \tag{A.2.18}$$

$$\begin{aligned} X_{21} &= A_{22}X_{21}(A_{11} - K_1C_1)^{\mathrm{T}} + A_{21}X_{11}(A_{11} - K_1C_1)^{\mathrm{T}} \\ &\quad + B_2 J (B_1 - K_1 D)^{\mathrm{T}} \end{aligned} \tag{A.2.19}$$

$$\begin{aligned} X_{22} &= A_{22}X_{22}A_{22}^{\mathrm{T}} + A_{21}X_{11}A_{21}^{\mathrm{T}} + A_{22}X_{21}A_{21}^{\mathrm{T}} + A_{21}^{\mathrm{T}}X_{21}^{\mathrm{T}}A_{22}^{\mathrm{T}} \\ &\quad + B_2 J B_2^{\mathrm{T}} - K_2 V K_2^{\mathrm{T}} \end{aligned} \tag{A.2.20}$$

where $K = (AXC^{\mathrm{T}} + BJD^{\mathrm{T}})V^{-1}$ is partitioned as

$$K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} (A_{11}X_{11}C_1^{\mathrm{T}} + B_1 J D^{\mathrm{T}})V^{-1} \\ \{(A_{21}X_{11} + A_{22}X_{21})C_1^{\mathrm{T}} + B_2 J D^{\mathrm{T}}\}V^{-1} \end{bmatrix}$$

Since $G(\sigma) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$, from the result in the case where (C, A) is observable, there exists a unique stabilizing solution X_{11} to the ARE of (A.2.17),(A.2.18). Hence $A_{11} - K_1 C_1$ is stable. Since A_{22} and $A_{11} - K_1 C_1$ are stable, it follows from (A.2.19) and (A.2.20) that X_{21} and X_{22} are uniquely determined for X_{11}. Furthermore, we have

$$A - KC = A - \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \begin{bmatrix} C_1 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} - K_1 C_1 & 0 \\ A_{21} - K_2 C_1 & A_{22} \end{bmatrix}$$

From the stability of A_{22} and $A_{11} - K_1 C_1$, $A - KC$ is also stable. Therefore, in the case where (C, A) is detectable, the ARE (2.3.1) also has a unique stabilizing solution X. ■

Appendix 2.2: Proof of Lemma 2.2

Since $V_{11} > 0$, the following identity holds.

$$V = \begin{bmatrix} I_q & 0 \\ V_{21}V_{11}^{-1} & I_p \end{bmatrix} \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} - V_{21}V_{11}^{-1}V_{21}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I_q & V_{11}^{-1}V_{21}^{\mathrm{T}} \\ 0 & I_p \end{bmatrix} \tag{A.2.21}$$

For the existence of a nonsingular W, it is necessary that $V_{22} - V_{21}V_{11}^{-1}V_{21}^{\mathrm{T}}$ is nonsingular.

We define

$$\begin{bmatrix} \widehat{W}_{11} & \widehat{W}_{12} \\ \widehat{W}_{21} & \widehat{W}_{22} \end{bmatrix} = \begin{bmatrix} I_q & 0 \\ -V_{21}V_{11}^{-1} & I_p \end{bmatrix} W$$

Then, from $WJ_{pq}W^{\mathrm{T}} = V$ and (A.2.21), we obtain

$$V_{11} = \widehat{W}_{11}\widehat{W}_{11}^{\mathrm{T}} - \gamma^2\widehat{W}_{12}\widehat{W}_{12}^{\mathrm{T}} \tag{A.2.22}$$

$$0 = \widehat{W}_{21}\widehat{W}_{11}^{\mathrm{T}} - \gamma^2\widehat{W}_{22}\widehat{W}_{12}^{\mathrm{T}} \tag{A.2.23}$$

$$V_{22} - V_{21}V_{11}^{-1}V_{21}^{\mathrm{T}} = \widehat{W}_{21}\widehat{W}_{21}^{\mathrm{T}} - \gamma^2\widehat{W}_{22}\widehat{W}_{22}^{\mathrm{T}} \tag{A.2.24}$$

Since $V_{11} > 0$ and (A.2.22) hold, $\widehat{W}_{11}$ is invertible. Then, from (A.2.23), we get $\widehat{W}_{21}^{\mathrm{T}} = \gamma^2\widehat{W}_{11}^{-1}\widehat{W}_{12}\widehat{W}_{22}^{\mathrm{T}}$. Substituting into (A.2.24) yields

$$V_{21}V_{11}^{-1}V_{21}^{\mathrm{T}} - V_{22} = \gamma^2\widehat{W}_{22}(I_p - N^{\mathrm{T}}N)\widehat{W}_{22}^{\mathrm{T}} \tag{A.2.25}$$

where $N = \gamma\widehat{W}_{11}^{-1}\widehat{W}_{12}$. Since $I_q - NN^{\mathrm{T}} > 0$ holds from (A.2.22), we get $I_p - N^{\mathrm{T}}N > 0$. Thus, it follows from (A.2.25) that $V_{21}V_{11}^{-1}V_{21}^{\mathrm{T}} - V_{22} \geq 0$. Since $V_{21}V_{11}^{-1}V_{21}^{\mathrm{T}} - V_{22}$ is invertible, we obtain

$$V_{21}V_{11}^{-1}V_{21}^{\mathrm{T}} - V_{22} > 0$$

Conversely, assume that $V_{21}V_{11}^{-1}V_{21}^{\mathrm{T}} - V_{22} > 0$ holds. Then it is easy to verify that

$$W = \begin{bmatrix} V_{11}^{1/2} & 0 \\ V_{21}V_{11}^{-1/2} & \gamma^{-1}(V_{21}V_{11}^{-1}V_{21}^{\mathrm{T}} - V_{22})^{1/2} \end{bmatrix} \tag{A.2.26}$$

is invertible and satisfies $WJ_{qp}W^{\mathrm{T}} = V$. ■

Appendix 2.3: Proof of Lemma 2.5

Although a proof is given in [14], we give a different proof.

Necessity: Under the assumption (A2), there exists a matrix $H \in \mathbf{R}^{n\times q}$ such that $A_H := A - HC$ is stable. We define

$$T_{\mathrm{f1}}(\sigma) = \left[\begin{array}{c|c} A_H & H \\ \hline L & 0 \end{array}\right]$$

We easily see that $T_{\mathrm{f1}}(\sigma)$ satisfies (S1) and (S2). Therefore, without loss of generality, we can assume that $T_{\mathrm{f}}(\sigma)$ has the form of

$$T_{\mathrm{f}}(\sigma) = T_{\mathrm{f1}}(\sigma) + T_{\mathrm{f}}'(\sigma) \tag{A.2.27}$$

where $T_f'(\sigma) \in \mathbf{RH}_\infty^{p\times q}$. We now define

$$T_{ed}^0(\sigma) = T_{zd}(\sigma) - T_{f1}(\sigma)T_{yd}(\sigma) = \left[\begin{array}{c|c} A_H & B_H \\ \hline L & 0 \end{array}\right] \in \mathbf{RH}_\infty^{p\times m}$$

where $B_H = B - HD$. Substituting (A.2.27) into (2.2.7) yields

$$\begin{aligned} T_{ed}(\sigma) &= T_{ed}^0(\sigma) - T_f'(\sigma)T_{yd}(\sigma) \\ &= \left[\begin{array}{c|c} A_H & B_H \\ \hline L & 0 \end{array}\right] - T_f'(\sigma) \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \end{aligned} \tag{A.2.28}$$

Using the matrix H, a left coprime factorization of $T_{yd}(\sigma)$ is given by

$$T_{yd}(\sigma) = X^{-1}(\sigma)Y(\sigma)$$

where

$$X(\sigma) = \left[\begin{array}{c|c} A_H & H \\ \hline -C & I_q \end{array}\right], \; Y(\sigma) = \left[\begin{array}{c|c} A_H & B_H \\ \hline C & D \end{array}\right]$$

From (A.2.28), we get

$$T_f'(\sigma)X^{-1}(\sigma)Y(\sigma) = T_{ed}(\sigma) - T_{ed}^0(\sigma) \in \mathbf{RH}_\infty^{p\times m} \tag{A.2.29}$$

Since $X(\sigma)$ and $Y(\sigma)$ are coprime over $\mathbf{RH}_\infty$, it follows from (A.2.29) that $T_f'(\sigma)X^{-1}(\sigma) \in \mathbf{RH}_\infty^{p\times q}$ holds. Therefore, by defining $Q(\sigma) = -T_f'(\sigma)X^{-1}(\sigma)$ and $T_{f2}(\sigma) = X(\sigma)$, we get the parametrization of (2.4.1).

Sufficiency: The stability of $T_f(\sigma)$ is obvious. If $T_f(\sigma)$ is expressed as in (2.4.1) and (2.4.2), then a straightforward calculation yields

$$T_{ed}(\sigma) = \left[\begin{array}{c|c} A_H & B_H \\ \hline L & 0 \end{array}\right] - Q(\sigma) \left[\begin{array}{c|c} A_H & B_H \\ \hline C & D \end{array}\right]$$

Since A_H is stable and $Q(\sigma) \in \mathbf{RH}_\infty^{p\times q}$, $T_{ed}(\sigma) \in \mathbf{RH}_\infty^{p\times m}$ holds. ■

Chapter 3

$\mathbf{H}_\infty$ Filtering with Boundary Constraints

1. Introduction

In the previous chapter, we have derived the parametrization of all solutions of the infinite-horizon $\mathbf{H}_\infty$ filtering problem for time-invariant systems. The free parameter of the $\mathbf{H}_\infty$ filter can be used to achieve an additional design specification as well as the $\mathbf{H}_\infty$ error bound. As an example of such multi-objective $\mathbf{H}_\infty$ filtering problems, we consider the $\mathbf{H}_\infty$ filtering problem with frequency constraints on the unit circle of the complex plane (boundary constraints).

If the system is subject to step or periodic disturbances, then the state estimates may be degraded by the biases or the periodic fluctuations due to these disturbances. In order to reject these undesirable effects, we need to impose boundary constraints such that the transfer functions from these disturbances to the error must be zero at certain points on the unit circle. Thus, the state estimation with boundary constraints is also important from the practical viewpoint.

It may be also noted that we can attenuate step and periodic disturbances by applying the observer design technique to the augmented system incorporating the state-space model of the disturbances. However, since the augmented system does not satisfy the assumption (A2) in Chapter 2, it is difficult to solve the $\mathbf{H}_\infty$ filtering problem with boundary constraints by the conventional observer design technique.

Therefore, in this chapter, based on the Nevanlinna-Pick interpolation technique [45], [48], we develop a method for adjusting the free parameter of the $\mathbf{H}_\infty$ filter so that the boundary constraints are satisfied. Moreover, we show that the resulting $\mathbf{H}_\infty$ filter is a linear function observer for the augmented system including the disturbance model. A numerical example also shows the applicability of the proposed design method.

2. Problem Formulation

We again consider the system of (2.2.1)–(2.2.3)

$$x_{k+1} = Ax_k + Bd_k$$
$$y_k = Cx_k + Dd_k$$
$$z_k = Lx_k$$

where $x_k \in \mathbf{R}^n$, $y_k \in \mathbf{R}^q$ and $d_k \in \mathbf{R}^m$ are the state vector, the measurement and the disturbance at time k, respectively. Moreover, $z_k \in \mathbf{R}^p$ is the vector to be estimated.

As in Chapter 2, we assume the following.

(A1) (C, A) is detectable.

(A2) $\mathrm{rank} \begin{bmatrix} A - e^{j\omega} I_n & B \\ C & D \end{bmatrix} = n + q, \quad \forall \omega \in \mathbf{R}$

Let $\widehat{z}_k$ be the estimate of z_k based on the measurement set $\{y_t |\ t \leq k\}$, and $T_f(\sigma)$ be the filter transfer matrix from y_k to $\widehat{z}_k$. The standard $\mathbf{H}_\infty$ filtering problem is the problem of finding a filter $T_f(\sigma)$ satisfying the following specifications:

(S1) $T_f(\sigma) \in \mathbf{RH}_\infty^{p\times q}$

(S2) $T_{ed}(\sigma) \in \mathbf{RH}_\infty^{p\times m}$

(S3) $\|T_{ed}\|_\infty < \gamma$ for a given constant $\gamma > 0$

where $T_{ed}(\sigma)$ is the transfer matrix from d_k to the estimation error $e_k := z_k - \widehat{z}_k$.

$$T_{ed}(\sigma) = \left[\begin{array}{c|c} A & B \\ \hline L & 0 \end{array}\right] - T_f(\sigma) \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

In this chapter, we wish to find a filter $T_f(\sigma)$ satisfying the boundary constraints on the unit circle in the complex plane in addition to the above specifications.

(S4) $T_{ew}^{(i)}(e^{j\omega_i}) = 0$ for given r different frequency points $\omega_i \in \mathbf{R}$ $(i = 1, \cdots, r)$

where $w_k^{(i)}$ is the column vector which consists of the ℓ_i entries of d_k associated with the i-th constraint, and $T_{ew}^{(i)}(\sigma)$ is the transfer matrix from $w_k^{(i)}$ to the estimation error e_k. Moreover, we define $v_k^{(i)}$ as the column vector which consists of the remaining $m - \ell_i$ entries of d_k, and let $T_{ev}^{(i)}(\sigma)$ be the transfer matrix from $v_k^{(i)}$ to e_k.

In many practical situations, the disturbance d_k may include step or periodic disturbances. If d_k contains these disturbances, then the estimates may be degraded by the biases or the periodic fluctuations. In order to remove these undesirable effects, it is required that the transfer matrix from the disturbances to the estimation error should be zero at certain points on the unit circle. To see this more specifically, let us consider the following example.

Example 3.1: We consider the system given by

$$
\begin{aligned}
x_{k+1} &= \begin{bmatrix} 5 & 0.5 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} d_k \\
y_k &= \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} d_k \\
z_k &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$
$$(m = 3,\ n = 3,\ p = 2,\ q = 2)$$

We denote the i-th element of d_k by $d_{i,k}$, $i = 1, 2, 3$, and assume that $d_{1,k}$ contains a step disturbance and $d_{3,k}$ contains both a step disturbance and a periodic disturbance with frequency $\omega_2 > 0$. We now define $w_k^{(i)}$ and $v_k^{(i)}$, $i = 1, 2$ $(r = 2)$, by

$$
w_k^{(1)} = \begin{bmatrix} d_{1,k} \\ d_{3,k} \end{bmatrix}, \quad v_k^{(1)} = d_{2,k}
$$
$$
w_k^{(2)} = d_{3,k}, \qquad v_k^{(2)} = \begin{bmatrix} d_{1,k} \\ d_{2,k} \end{bmatrix}
$$

Further, we define $T_{ew}^{(i)}(\sigma)$ and $T_{ev}^{(i)}(\sigma)$ as the transfer matrices from $w_k^{(i)}$ and $v_k^{(i)}$ to e_k, respectively.

Since $w_k^{(1)}$ contains step disturbances, we assume that $v_k^{(1)} \in \mathbf{L}_2$ and

$$w_k^{(1)} = \begin{cases} a_1 & 0 \le k \\ 0 & k < 0 \end{cases}$$

where $a_1 \in \mathbf{R}^2$ is an arbitrary constant vector. Then, by the final value theorem, we get

$$\begin{aligned} \lim_{k\to\infty} e_k &= \lim_{\sigma\to 1}(1-\sigma^{-1})T_{ew}^{(1)}(\sigma)w^{(1)}(\sigma) \\ &= \lim_{\sigma\to\infty}(1-\sigma^{-1})T_{ew}^{(1)}(\sigma)a_1\frac{1}{1-\sigma^{-1}} \\ &= T_{ew}^{(1)}(1)a_1 \end{aligned}$$

Thus, in order to reject the bias due to the step disturbance, we need to design $T_f(\sigma)$ so that $T_{ew}^{(1)}(1) = 0$ hold. This implies that $T_{ew}^{(1)}(e^{j\omega_1}) = 0$ with $\omega_1 = 0$.

Next, we consider the effect of the periodic disturbance. Allowing $w_k^{(2)}$ to take a complex value for simplicity, we assume that $w_k^{(2)} = a_2\exp(j\omega_2 k)$ for some constant $a_2 \in \mathbf{R}$ and $v_k^{(2)} \in \mathbf{L}_2$. Then, the steady-state response of e_k is given by

$$e_k = T_{ew}^{(2)}(e^{j\omega_2})a_2 e^{j\omega_2 k}$$

Hence, in order to remove the periodic fluctuation due to the periodic disturbance, we require that $T_{ew}^{(2)}(e^{j\omega_2}) = 0$ should be satisfied.

3. Boundary Constraints

Assume that the conditions (a) and (b) of Theorem 2.1 hold, and that the parametrization of all $T_f(\sigma)$ satisfying (S1)–(S3) is given by (2.4.21) and (2.4.22), where $U(\sigma)$ is an arbitrary transfer matrix in $\mathbf{BH}_\infty^{p\times q}$, and where $K = [K_1 \ \ K_2] \in \mathbf{R}^{n\times(q+p)}$, A_K and W are defined by (2.4.16), (2.4.17) and (2.4.13), respectively. Then we see that $T_{ed}(\sigma)$ is given by

$$T_{ed}(\sigma) = -\Sigma_2^{-1}(\sigma)\Sigma_1(\sigma) \tag{3.3.1}$$

where

$$\begin{aligned} \begin{bmatrix} \Sigma_1(\sigma) & \Sigma_2(\sigma) \end{bmatrix} &= \begin{bmatrix} U(\sigma) & -I_p \end{bmatrix}\Theta(\sigma) \\ &= \begin{bmatrix} U\Theta_{11} - \Theta_{21} & U\Theta_{12} - \Theta_{22} \end{bmatrix} \end{aligned} \tag{3.3.2}$$

and $\Theta(\sigma)$ is defined by (2.4.23). We here define $\Sigma_{11}^{(i)}(\sigma)$ and $\Sigma_{12}^{(i)}(\sigma)$ as the transfer matrices which consist of the column vectors of $\Sigma_1(\sigma)$ corresponding to $w_k^{(i)}$ and $v_k^{(i)}$, respectively. In other words, we define $\Sigma_{11}^{(i)}(\sigma)$ and $\Sigma_{12}^{(i)}(\sigma)$ so that

$$T_{ew}^{(i)}(\sigma) = -\Sigma_2^{-1}\Sigma_{11}^{(i)}, \quad T_{ev}^{(i)}(\sigma) = -\Sigma_2^{-1}\Sigma_{12}^{(i)} \tag{3.3.3}$$

The following theorem shows that the boundary constraints can be expressed in terms of $\Sigma_{11}^{(i)}(\sigma)$.

Theorem 3.1: *Suppose that the conditions (a) and (b) of Theorem 2.1 hold, and that the $\mathbf{H}_\infty$ filter satisfying (S1)–(S3) is given by Theorem 2.2. Then, for a given frequency point $\omega_i \in \mathbf{R}$, $T_{ew}^{(i)}(e^{j\omega_i}) = 0$ holds if and only if $\Sigma_{11}^{(i)}(e^{j\omega_i}) = 0$ holds.*

Proof: *(Necessity)* Since $\Sigma_2(\sigma) \in \mathbf{RH}_\infty^{p\times q}$ holds from $U(\sigma) \in \mathbf{RH}_\infty^{p\times q}$ and $\Theta(\sigma) \in \mathbf{RH}_\infty^{(q+p)\times(m+p)}$, $\Sigma_2^{-1}(\sigma)$ does not have a zero at $\sigma = e^{j\omega_i}$. Hence $\Sigma_{11}^{(i)}(e^{j\omega_i}) = 0$ is necessary in order that $T_{ew}^{(i)}(e^{j\omega_i}) = 0$ should hold.

(Sufficiency) Assume that $\Sigma_{11}^{(i)}(e^{j\omega_i}) = 0$. Then, from (3.3.1), it suffices to show that $\Sigma_2(\sigma)$ does not have a zero at $\sigma = e^{j\omega_i}$. Since $\Theta(\sigma)$ is (J_{mp}, J_{qp})-lossless, we get

$$\Theta(e^{j\omega_i})J_{mp}\Theta^{\mathrm{H}}(e^{j\omega_i}) = J_{qp}$$

Pre-multiplying this by $[U(e^{j\omega_i}) \ \ I_p]$ and post-multiplying by $[U(e^{j\omega_i}) \ \ I_p]^{\mathrm{H}}$ yield

$$\gamma^2 I_p - UU^{\mathrm{H}} = \gamma^2\Sigma_2\Sigma_2^{\mathrm{H}} - \Sigma_1\Sigma_1^{\mathrm{H}}, \quad \sigma = e^{j\omega_i}$$

where $U(\sigma) \in \mathbf{RH}_\infty^{p\times q}$ is a free parameter satisfying $\|U\|_\infty < \gamma$. It follows from $\Sigma_{11}^{(i)}(e^{j\omega_i}) = 0$ that

$$\gamma^2 I_p - UU^{\mathrm{H}} = \gamma^2\Sigma_2\Sigma_2^{\mathrm{H}} - \Sigma_{12}^{(i)}\Sigma_{12}^{(i)\mathrm{H}}, \quad \sigma = e^{j\omega_i} \tag{3.3.4}$$

It may be noted that the left-hand side of this equation is positive define since $\|U\|_\infty < \gamma$. If $\Sigma_2(\sigma)$ has a zero at $\sigma = e^{j\omega_i}$, there exists a nonzero vector η such that $\eta^{\mathrm{H}}\Sigma_2(e^{j\omega_i}) = 0$. Then we see from (3.3.4) that

$$\eta^{\mathrm{H}}(\gamma^2 I_p - UU^{\mathrm{H}})\eta = -\eta^{\mathrm{H}}\Sigma_{12}^{(i)}\Sigma_{12}^{(i)\mathrm{H}}\eta \le 0, \quad \sigma = e^{j\omega_i}$$

This contradicts the positive definiteness of $\gamma^2 I_p - UU^{\mathrm{H}}$. Therefore, $\Sigma_2(\sigma)$ does not have a zero at $\sigma = e^{j\omega_i}$. ■

We here define $\Theta_{111}^{(i)}(\sigma)$ and $\Theta_{112}^{(i)}(\sigma)$ as the transfer matrices which consist of the column vectors of $\Theta_{11}(\sigma)$ corresponding to $w_k^{(i)}$ and $v_k^{(i)}$, respectively. We similarly define $\Theta_{211}^{(i)}(\sigma)$ and $\Theta_{2112}^{(i)}(\sigma)$ for $\Theta_{21}(\sigma)$, so that

$$\Sigma_{11}^{(i)}(\sigma) = U(\sigma)\Theta_{111}^{(i)}(\sigma) - \Theta_{211}^{(i)}(\sigma) \tag{3.3.5}$$

$$\Sigma_{12}^{(i)}(\sigma) = U(\sigma)\Theta_{112}^{(i)}(\sigma) - \Theta_{212}^{(i)}(\sigma) \tag{3.3.6}$$

Since $\Sigma_{11}^{(i)}(\sigma)$ is affine with respect to $U(\sigma)$, from Theorem 3.1, finding a free parameter $U(\sigma)$ satisfying (S4) reduces to the following interpolation problem.

Interpolation Problem: For given r different frequency points ω_i $(i = 1, \cdots, r)$, find a transfer matrix $U(\sigma) \in \mathbf{BH}_{\infty}^{p\times q}$ such that

$$U(e^{j\omega_i}) = U_i \quad (i = 1, \cdots, r) \tag{3.3.7}$$

where $U_i \in \mathbf{C}^{p\times q}$ is a solution of the linear matrix equation

$$U_i\Theta_{111}^{(i)}(e^{j\omega_i}) = \Theta_{211}^{(i)}(e^{j\omega_i}) \tag{3.3.8}$$

A solution of this interpolation problem is given by the following theorem.

Theorem 3.2: *Suppose that the conditions (a) and (b) of Theorem 2.1 hold, and that $\Theta(\sigma)$ is defined by (2.4.23). Then there exist a matrix U_i satisfying (3.3.8) if and only if*

$$\operatorname{Ker}\Theta_{111}^{(i)}(e^{j\omega_i}) \subseteq \operatorname{Ker}\Theta_{211}^{(i)}(e^{j\omega_i}) \tag{3.3.9}$$

If such a matrix U_i exists for $i = 1, \cdots, r$, then a necessary and sufficient condition for the existence of a free parameter $U(\sigma) \in \mathbf{BH}_{\infty}^{p\times q}$ satisfying (3.3.7) is that

$$\|\Theta_{211}^{(i)}(e^{j\omega_i})\Theta_{111}^{(i)\#}(e^{j\omega_i})\| < \gamma \qquad (i = 1, \cdots, r) \tag{3.3.10}$$

Proof: From Lemma A.3.1(i), there exists a matrix U_i satisfying (3.3.8) if and only if (3.3.9) holds. In this case, from Lemma A.3.1 (iii), the minimum-norm solution among all solutions to (3.3.8) is given by

$$U_i = \Theta_{211}^{(i)}(e^{j\omega_i})\Theta_{111}^{(i)\#}(e^{j\omega_i}) \tag{3.3.11}$$

Since $\|U\|_\infty < \gamma$ holds, for the existence of a matrix $U(\sigma) \in \mathbf{BH}_\infty^{p\times q}$ satisfying (3.3.7), there must exist a matrix satisfying $\|U_i\| < \gamma$ among the solutions of (3.3.8). This implies that (3.3.10) must hold.

Conversely, we assume that the conditions (3.3.9) and (3.3.10) hold. Then there exists a matrix U_i satisfying (3.3.8) with $\|U_i\| < \gamma$. If $\|U_i\| < \gamma$ holds for $i = 1, \cdots, r$, then the existence of $U(\sigma) \in \mathbf{BH}_\infty^{p\times q}$ satisfying (3.3.7) is guaranteed by Lemma A.3.2. ■

In summary, the $\mathbf{H}_\infty$ filter $T_f(\sigma)$ satisfying (S1)–(S4) can be obtained by the following design procedure.

Step 1: Check if the conditions (a) and (b) of Theorem 2.1 hold or not. If not, stop.

Step 2: Solve the ARE of (2.4.6),(2.4.7), and obtain $\Theta(\sigma)$ from (2.4.23).

Step 3: Check if the conditions (3.3.9) and (3.3.10) hold or not. If not, stop.

Step 4: Obtain U_i from (3.3.11), and find a $U(\sigma) \in \mathbf{RH}_\infty^{p\times q}$ satisfying $\|U\|_\infty < \gamma$ and (3.3.8) using the matrix-valued Nevanlinna-Pick algorithm [48].

Step 5: Obtain $T_f(\sigma)$ from (2.4.21),(2.4.22) in Theorem 2.2.

In general, the conditions (3.3.9),(3.3.10) depend on the parameter γ. However, if A has no eigenvalues on the unit circle, then we can obtain a condition equivalent to (3.3.9), which is independent of γ.

Lemma 3.1: *Suppose that A has no eigenvalues on the unit circle. Then, the following condition is equivalent to (3.3.9) in Theorem 3.2:*

$$\operatorname{Ker} T_{yw}^{(i)}(e^{j\omega_i}) \subseteq \operatorname{Ker} T_{zw}^{(i)}(e^{j\omega_i}) \tag{3.3.12}$$

where $T_{yw}^{(i)}(\sigma)$ and $T_{zw}^{(i)}(\sigma)$ are the transfer matrices from $w_k^{(i)}$ to y_k and z_k, respectively.

Proof: Note that the eigenvalues of A are the invariant zeros of $\Omega(\sigma)$ of Theorem 2.2. It follows that, if A has no eigenvalues on the unit circle, there exists a matrix $\Gamma_i \in \mathbf{C}^{p\times q}$ such that

$$\Gamma_i = -\{U_i\Omega_{12}(e^{j\omega_i}) - \Omega_{22}(e^{j\omega_i})\}^{-1}\{U_i\Omega_{11}(e^{j\omega_i}) - \Omega_{21}(e^{j\omega_i})\} \tag{3.3.13}$$

Conversely, if a matrix Γ_i is given, we can get U_i by

$$U_i = -\{\Gamma_i\bar{\Omega}_{12}(e^{j\omega_i}) - \bar{\Omega}_{22}(e^{j\omega_i})\}^{-1}\{\Gamma_i\bar{\Omega}_{11}(e^{j\omega_i}) - \bar{\Omega}_{21}(e^{j\omega_i})\} \tag{3.3.14}$$

where

$$\begin{bmatrix} \bar{\Omega}_{11} & \bar{\Omega}_{12} \\ \bar{\Omega}_{21} & \bar{\Omega}_{22} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}^{-1}.$$

Thus, the mapping from U_i to Γ_i is bijective. Furthermore, a simple calculation of state-space data yields

$$\begin{bmatrix} \Theta_{111}^{(i)} \\ \Theta_{211}^{(i)} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \begin{bmatrix} T_{yw}^{(i)} \\ T_{zw}^{(i)} \end{bmatrix} \tag{3.3.15}$$

Hence, we see from (3.3.8),(3.3.13) and (3.3.15) that the existence of U_i satisfying (3.3.8) is equivalent to the existence of Γ_i satisfying

$$\Gamma_i T_{yw}^{(i)}(e^{j\omega_i}) = T_{zw}^{(i)}(e^{j\omega_i}) \tag{3.3.16}$$

Therefore, by Lemma A.3.1 of Appendix 3.1, the condition (3.3.9) is equivalent to (3.3.12).
■

4. Relation to Linear Function Observer Theory

In the previous section, we discussed the boundary constraints from the viewpoint of the zeros of transfer matrices. Therefore, the relationship between the structure of the resulting $\mathbf{H}_\infty$ filter and the disturbance model is not clear. On the other hand, it is well known that the linear function observer theory can be applied to the state estimation for the augmented system incorporating the disturbance model. In this section, we will clarify the relationship between the resulting $\mathbf{H}_\infty$ filter and the conventional linear function observer by showing that the resulting $\mathbf{H}_\infty$ filter is a linear function observer for the augmented system.

Hereafter, we only consider the step disturbance ($r = 1$, $\omega_1 = 0$) for simplicity. We also assume without loss of generality that $d_k = [w_k^{(1)\mathrm{T}} \ v_k^{(1)\mathrm{T}}]^\mathrm{T}$ and

$$B = [B_1 \ \ B_2], \quad D = [D_1 \ \ D_2]$$

accordingly. Then the system of (2.2.1),(2.2.2) is expressed as

$$x_{k+1} = Ax_k + B_1 w_k^{(1)} + B_2 v_k^{(1)} \tag{3.4.1}$$

$$y_k = Cx_k + D_1 w_k^{(1)} + D_2 v_k^{(1)} \tag{3.4.2}$$

Since $w_k^{(1)}$ is a step function, the disturbance model is given by

$$w_{k+1}^{(1)} = w_k^{(1)} \tag{3.4.3}$$

We assume that the conditions (i),(ii) of Theorem 2.1 hold. We also assume without loss of generality that a matrix W satisfying (2.4.13) has the form

$$W = \begin{bmatrix} W_{11} & 0 \\ W_{21} & W_{22} \end{bmatrix}$$

and define

$$W_1 = \begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 \\ W_{22} \end{bmatrix}$$

In the case where $r = 1$ and $\omega_1 = 0$, the interpolation condition of (3.3.7) and (3.3.8) can be given by

$$U(e^{j\omega_1}) = U(1) = U_1$$

where

$$U_1 = \Theta_{211}^{(1)}(1)\Theta_{111}^{(1)\#}(1)$$

Since $U_1 \in \mathbf{R}^{p\times q}$, we can choose the free parameter $U(\sigma)$ as

$$U(\sigma) \equiv U_1 = \Theta_{211}^{(1)}(1)\Theta_{111}^{(1)\#}(1) \tag{3.4.4}$$

From the above discussion, an $\mathbf{H}_\infty$ filter satisfying (S1)-(S4) is given by

$$T_{\mathrm{f}}(\sigma) = \left[\begin{array}{c|c} A - \widehat{K}C & \widehat{K} \\ \hline L - \widehat{M}C & \widehat{M} \end{array}\right] \tag{3.4.5}$$

$$\widehat{K} = KW_1W_{11}^{-1} + K_2W_{22}UW_{11}^{-1} \tag{3.4.6}$$

$$\widehat{M} = W_{21}W_{11}^{-1} + W_{22}UW_{11}^{-1} \tag{3.4.7}$$

where $K = [K_1 \quad K_2]$ is defined by (2.4.16).

Since $A - \widehat{K}C$ is stable, we can define

$$T := \begin{bmatrix} I_n & -\{I_n - (A - \widehat{K}C)\}^{-1}(B_1 - \widehat{K}D_1) \end{bmatrix}$$

Simple calculation yields

$$T \begin{bmatrix} A & B_1 \\ 0 & I_{\ell_1} \end{bmatrix} - (A - \widehat{K}C)T = \widehat{K} \begin{bmatrix} C & D_1 \end{bmatrix} \tag{3.4.8}$$

Also, since $T_{ew}^{(1)}(1) = 0$, we get

$$(L - \widehat{M}C)\{I_n - (A - \widehat{K}C)\}^{-1}(B_1 - \widehat{K}D_1) - \widehat{M}D_1 = 0$$

That is,

$$(L - \widehat{M}C)T + \widehat{M} \begin{bmatrix} C & D_1 \end{bmatrix} = \begin{bmatrix} L & 0 \end{bmatrix} \tag{3.4.9}$$

It therefore follows from (3.4.8),(3.4.9) and Lemma A.3.3 that $T_f(\sigma)$ of (3.4.5) is a linear function observer for the augmented system

$$\begin{bmatrix} x_{k+1} \\ w_{k+1}^{(1)} \end{bmatrix} = \begin{bmatrix} A & B_1 \\ 0 & I_{\ell_1} \end{bmatrix} \begin{bmatrix} x_k \\ w_k^{(1)} \end{bmatrix} + \begin{bmatrix} B_2 \\ 0 \end{bmatrix} v_k^{(1)}$$

$$y_k = \begin{bmatrix} C & D_1 \end{bmatrix} \begin{bmatrix} x_k \\ w_k^{(1)} \end{bmatrix} + D_2 v_k^{(1)}$$

$$z_k = \begin{bmatrix} L & 0 \end{bmatrix} \begin{bmatrix} x_k \\ w_k^{(1)} \end{bmatrix}$$

5. Numerical Example

In this section, we consider Example 3.1 again. The infimum of γ satisfying the conditions (a),(b) of Theorem 2.1 is $\gamma = 3.112$. Thus, in the following, we take $\gamma = 3.5$. In this case, the stabilizing solution $P \geq 0$ to the ARE of (2.4.6),(2.4.7) is

$$P = \begin{bmatrix} 22.375 & -12.976 & -10.581 \\ -12.976 & 62.889 & 147.537 \\ -10.581 & 147.537 & 373.886 \end{bmatrix}$$

Also, a matrix W satisfying (2.4.13) is

$$W = \begin{bmatrix} 14.933 & -0.181 & 0 & 0 \\ -0.181 & 4.831 & 0 & 0 \\ 7.341 & 2.220 & 0.980 & -0.028 \\ 19.033 & -1.477 & -0.028 & 0.476 \end{bmatrix}$$

Since $w_k^{(1)}$ contains the step disturbance, $\omega_1 = 0$ for the constraint (S4). We also take $\omega_2 = \pi/4$ by assuming that the periodic disturbance included in $w_k^{(2)}$ has the frequency $\pi/4$. Note that the central filter ($U(\sigma) = 0$) does not satisfy (S4) for this example.

In order to find a free parameter satisfying (S4), we consider the interpolation problem in Section 3.3. Let U_i be given by (3.3.11), and let $\varepsilon_0 = 0.02$, where ε_0 is the parameter which reduces the interpolation problem on the unit circle to a usual Nevanlinna-Pick problem (see Appendix 3.2). Then, by applying the matrix-valued Nevanlinna-Pick algorithm [48], one of the solutions to the interpolation problem on the unit circle is given by the 8th-order transfer matrix

$$
\begin{aligned}
U(\sigma) &= \frac{1}{d(\sigma)} \begin{bmatrix} n_{11}(\sigma) & n_{12}(\sigma) \\ n_{21}(\sigma) & n_{21}(\sigma) \end{bmatrix} \\
d(\sigma) &= \sigma^8 - 6.206\sigma^7 + 17.708\sigma^6 - 30.322\sigma^5 + 33.997\sigma^4 \\
&\quad -25.506\sigma^3 + 12.485\sigma^2 - 3.636\sigma + 0.480 \\
n_{11}(\sigma) &= 0.211\sigma^8 - 1.335\sigma^7 + 3.876\sigma^6 - 6.736\sigma^5 + 7.652\sigma^4 \\
&\quad -5.820\sigma^3 + 2.875\sigma^2 - 0.846\sigma + 0.113 \\
n_{21}(\sigma) &= 0.928\sigma^8 - 5.421\sigma^7 + 14.624\sigma^6 - 23.657\sigma^5 + 24.921\sigma^4 \\
&\quad -17.395\sigma^3 + 7.794\sigma^2 - 2.022\sigma + 0.228 \\
n_{12}(\sigma) &= 0.317\sigma^8 - 1.985\sigma^7 + 5.696\sigma^6 - 9.772\sigma^5 + 10.937\sigma^4 \\
&\quad -8.157\sigma^3 + 3.952\sigma^2 - 1.134\sigma + 0.147 \\
n_{22}(\sigma) &= 0.188\sigma^8 - 1.392\sigma^7 + 4.545\sigma^6 - 8.674\sigma^5 + 10.661\sigma^4 \\
&\quad -8.676\sigma^3 + 4.579\sigma^2 - 1.437\sigma + 0.205
\end{aligned}
$$

For the resulting $\mathbf{H}_\infty$ filter, the values of $T_{ed}(\sigma)$ for $\sigma = 1, e^{j\pi/4}$ are

$$
\begin{aligned}
T_{ed}(1) &= \begin{bmatrix} 0.000 & -0.467 & 0.000 \\ 0.000 & 0.600 & 0.000 \end{bmatrix} \\
T_{ed}(e^{\frac{\pi}{4}j}) &= \begin{bmatrix} -0.001 + 0.042j & -0.415 + 0.069j & 0.000 \\ -0.470 - 1.243j & -0.240 - 2.812j & 0.000 \end{bmatrix}
\end{aligned}
$$

This shows that the condition (S4) is satisfied, that is, the transfer functions from $[d_{1,k}\ \ d_{3,k}]^{\mathrm{T}}$ and $d_{3,k}$ to e_k are zero at frequency $\omega = 0$ and $\pi/4$, respectively.

Fig. 3.1 shows the singular value (SV) plots of $T_{ed}(\sigma)$ for the central and proposed $\mathbf{H}_\infty$ filters. We see from the figure that the $\mathbf{H}_\infty$ error bound $\|T_{ed}\|_\infty < 3.5$ is achieved by both filters. It may be noted that, due to the constraint (S4), the SV plot for the proposed design method has a notch at frequency $\pi/4 \cong 0.785$ and has small singular values at low frequency band.

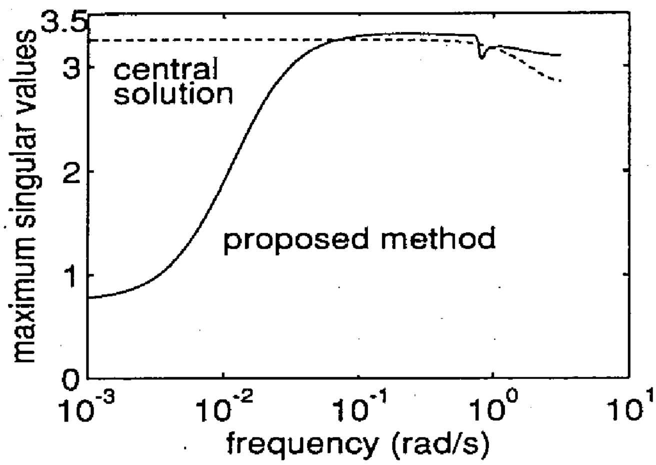


Fig. 3.1: Singular value plots of $T_{ed}(\sigma)$

A simulation result is also given in Fig. 3.2, where $e_k = [\,e_{1,k} \;\; e_{2,k}\,]^{\mathrm{T}}$. In order to see the effects of the step and periodic disturbances on the estimation errors, we give the disturbance d_k as follows:

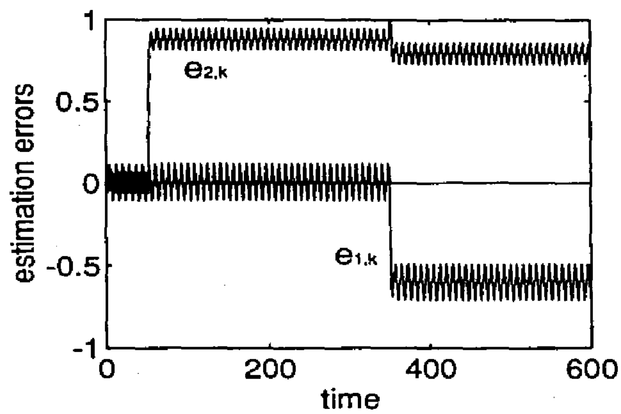
$$
\begin{aligned}
d_{1,k} &= \begin{cases} 0 & 0 \le k \le 50 \\ 0.5 & 50 < k \end{cases} \\
d_{2,k} &= 0 \qquad 0 \le k \\
d_{3,k} &= \begin{cases} 0.2\cos(\frac{\pi}{4}k) & 0 \le k \le 350 \\ 1 + 0.2\cos(\frac{\pi}{4}k) & 350 < k \end{cases}
\end{aligned}
$$

The initial states of the system and the filters are all set to zero. Fig. 3.2 (a) shows that the performance of the central $\mathbf{H}_\infty$ filter is degraded by the bias and the periodic fluctuations due to the step and periodic disturbances. On the contrary, in Fig. 3.2 (b), the estimation errors of the proposed $\mathbf{H}_\infty$ filter asymptotically converge to zero even in the presence of the step and periodic disturbances.

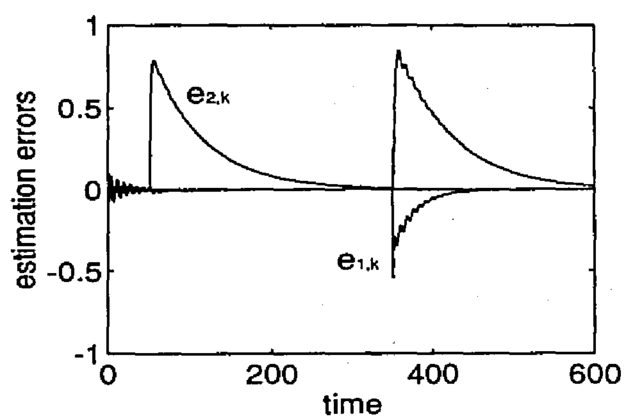
6. Concluding Remarks

In this chapter, we have proposed a design method of an $\mathbf{H}_\infty$ filter so that the constraints on the unit circle is satisfied. By this method, we can reject the undesirable effects due to the step or periodic disturbances.

We have also shown the relationship between the state-space model of the disturbance and the structure of the proposed $\mathbf{H}_\infty$ filter in the case where the disturbance is step function.



(a) Central solution



(b) Proposed method

Fig. 3.2: Simulation results

Appendix 3.1: Linear Matrix Equation

Lemma A.3.1: Consider the linear matrix equation

$$XB = C \tag{A.3.1}$$

for given constant matrices $B \in \mathbf{C}^{m\times n}$ and $C \in \mathbf{C}^{p\times n}$.

(i) There exists a matrix $X \in \mathbf{C}^{p\times m}$ satisfying (A.3.1) if and only if

$$\operatorname{Ker} B \subseteq \operatorname{Ker} C \qquad (\text{ equivalently, } \quad CB^{\#}B = C\,) \tag{A.3.2}$$

(ii) If (A.3.2) holds, the set of all solutions X is given by

$$X = CB^{\#} + Q(I_m - BB^{\#}) \tag{A.3.3}$$

(iii) The minimum-norm solution whose norm is minimal among all solutions is

$$X = CB^{\#} \tag{A.3.4}$$

Proof: See, for example, the reference [44]. ■

Appendix 3.2: Interpolation Problem on the Unit Circle

Lemma A.3.2: For given r different frequency points $\omega_i \in \mathbf{R}$ $(i = 1, \cdots, r)$, there exists a rational transfer matrix $X(\sigma) \in \mathbf{H}_{\infty}^{p\times q}$ satisfying $\|X\|_{\infty} < 1$ and

$$X(e^{j\omega_i}) = X_i \quad (i = 1, \cdots, r) \tag{A.3.5}$$

if and only if

$$\|X_i\| < 1, \quad (i = 1, \cdots, r) \tag{A.3.6}$$

Proof: The lemma is the discrete-time equivalent of Lemma B in [45]. Hence, the proof almost follows the line of the proof in [45].

(Necessity) Obvious.

(Sufficiency) We assume that (A.3.6) holds. We now define $\sigma_i(\varepsilon) = e^{\varepsilon + j\omega_i}$ for a small scalar $\varepsilon > 0$. Note that $\|\sigma_i(\varepsilon)\| > 1$ since $\varepsilon > 0$. We consider a Pick matrix [48] given by

$$P(\varepsilon) = \{P_{k\ell}(\varepsilon)\} \tag{A.3.7}$$

$$P_{k\ell}(\varepsilon) = \frac{I_p - X_k^{\mathrm{H}} X_\ell}{1 - \bar{\lambda}_k \lambda_\ell} = \frac{I_p - X_k^{\mathrm{H}} X_\ell}{1 - e^{-2\varepsilon - j(\omega_k - \omega_\ell)}} \tag{A.3.8}$$

where $\lambda_i = \sigma_i^{-1}$ and $\bar{\lambda}$ denotes the complex conjugate of λ. Since $\varepsilon > 0$ and (A.3.6) hold, we get $P_{kk}(\varepsilon) > 0$. Furthermore, as $\varepsilon \to 0$, $\|P_{kk}(\varepsilon)\|$ becomes arbitrarily large, whereas

$$\|P_{k\ell}(\varepsilon)\| \to \left\| \frac{I_p - X_k^{\mathrm{H}} X_\ell}{1 - e^{-j(\omega_k - \omega_\ell)}} \right\| < \infty, \quad k \neq \ell$$

Hence there exists a constant $\varepsilon_0 > 0$ such that $P(\varepsilon_0) > 0$ holds. This implies from the Nevanlinna-Pick theorem that there exists a rational transfer matrix $Z(\sigma) \in \mathbf{H}_\infty^{p \times q}$ satisfying $\|Z\|_\infty < 1$ and

$$Z(\sigma_i(\varepsilon_0)) = X_i \quad (i = 1, \cdots, r)$$

If we define $X(\sigma) := Z(e^{\varepsilon_0}\sigma)$ for such a $Z(\sigma)$, then $X(\sigma)$ satisfies $\|X\|_\infty < 1$ and the original interpolation condition (A.3.5). ■

Remark 3.1: $Z(\sigma)$ can be computed by the matrix-valued Nevanlinna-Pick algorithm [48].

Remark 3.2: In Lemma A.3.2, the existence of a "real" rational $X(\sigma)$ is not guaranteed. In order to obtain a real rational $X(\sigma)$, we need to impose additional conditions which are complex conjugates of the original conditions.

$$X(e^{-j\omega_i}) = \bar{X}_i \quad (i = 1, \cdots, r)$$

In this case, if we get a complex rational solution $X_0(\sigma)$, then a real rational solution $X(\sigma)$ is given by

$$X(\sigma) = \frac{1}{2}\{X_0(\sigma) + \overline{X_0(\bar{\sigma})}\}$$

Appendix 3.3: Linear Function Observer

Let us consider the system described by

$$x_{k+1} = Ax_k + Eu_k \tag{A.3.9}$$

$$y_k = Cx_k \tag{A.3.10}$$

where x_k, y_k and u_k is the state, measurement and the known control input.

We wish to estimate the linear function of the state variables defined by

$$z_k = Lx_k \tag{A.3.11}$$

Lemma A.3.3: Consider the system given by

$$\xi_{k+1} = \widehat{A}\xi_k + \widehat{B}y_k + \widehat{E}u_k \tag{A.3.12}$$

$$\widehat{z}_k = \widehat{C}\xi_k + \widehat{D}y_k \tag{A.3.13}$$

If the following conditions hold, the system of (A.3.12),(A.3.13) is a linear function observer for the system (A.3.9)–(A.3.11), i.e. $\widehat{z}_k \to z_k$ as $k \to \infty$.

(i) $\widehat{A}$: stable

(ii) There exists a constant matrix T satisfying

$$TA - \widehat{A}T = \widehat{B}C \tag{A.3.14}$$

$$\widehat{C}T + \widehat{D}C = L \tag{A.3.15}$$

$$\widehat{E} = TE \tag{A.3.16}$$

Proof: See the reference [10]. ■

Chapter 4

$\mathbf{H}_\infty$ Algebraic Riccati Equation and Parametrization of All $\mathbf{H}_\infty$ Filters

1. Introduction

Algebraic Riccati equations (ARE) play very important roles in the state-space solutions of many control and estimation problems. This chapter is concerned with the ARE related to the infinite-horizon $\mathbf{H}_\infty$ filtering problem and its application to the analysis of the $\mathbf{H}_\infty$ filter.

In the $\mathbf{H}_\infty$ filtering problem, we design a state estimator so that the $\mathbf{L}_2$ induced norm ($\mathbf{H}_\infty$ norm) of the error system is smaller than the prescribed bound γ. It has been shown that a necessary and sufficient condition for the existence of a solution to this problem is that an $\mathbf{H}_\infty$ ARE has a positive semi-definite stabilizing solution for which a certain matrix must be positive definite [40],[51],[52].

The $\mathbf{H}_\infty$ AREs arising in the $\mathbf{H}_\infty$ control and estimation problems have been extensively examined. For the continuous-time case, Hewer [22] and Gahinet [13] have shown that the stabilizing solution of the continuous-time $\mathbf{H}_\infty$ ARE is monotonically non-increasing convex function of γ, and the behavior at the optimum is considered by Gahinet [13]. A recursive method for obtaining the solution of the discrete-time $\mathbf{H}_\infty$ ARE and some related results have been given by Stoorvogel and Weeren [43]. It may be also noted that the existence condition of a stabilizing solution to the ARE of general type is considered based on the Popov function by Ionescu and Weiss [23]. In this paper, we will derive the infimum of γ for which a stabilizing solution to the discrete-time $\mathbf{H}_\infty$ ARE exists, and show that the positive semi-definite stabilizing solution has the monotonicity and convexity properties for γ, which are discrete-time counterparts of the results in [13],[22].

Since the state-space realization of the $\mathbf{H}_\infty$ filter is given in terms of the stabilizing solution of the $\mathbf{H}_\infty$ ARE, the performance of the $\mathbf{H}_\infty$ filter depends on the stabilizing solution. Therefore, the analyses of the $\mathbf{H}_\infty$ARE are very important. A relationship between the performance of the central $\mathbf{H}_\infty$ filter and the prescribed bound γ has been examined based on the monotonicity of the $\mathbf{H}_\infty$ RDE for the time-varying case by the authors (see Chapter 6). Also, multi-objective filter design problems including $\mathbf{H}_2/\mathbf{H}_\infty$ filtering problem [20],[27] aim at achieving an additional design specification by using the free parameter contained in the $\mathbf{H}_\infty$ filter. Thus, the performance for the additional specification depends on the size of the region where the free parameter ranges. Motivated by this observation, we will investigate the change of the size of this region with respect to the variation of γ based on the above properties of the $\mathbf{H}_\infty$ ARE. Such analyses of the $\mathbf{H}_\infty$ filter will provide a guideline for designing an $\mathbf{H}_\infty$ filter.

2. Algebraic Riccati Equation

In this section, we will give some results related to the stabilizing solution P.

Similarly to the previous chapters, the following two conditions are assumed for the system of (2.2.1)–(2.2.3).

(A1) (C, A) is detectable.

(A2) $\operatorname{rank} \begin{bmatrix} A - e^{j\omega} I_n & B \\ C & D \end{bmatrix} = n + q, \quad \forall \omega \in \mathbf{R}$

For simplicity of discussion, we hereafter assume the following condition.

(A3) $R := DD^{\mathrm{T}} > 0$

This assumption implies that all elements of y_k are degraded by the disturbance d_k. Such a situation can be found in many practical applications. It may be also noted that, in the case where D is degenerated, the $\mathbf{H}_\infty$ filtering problem for the system (2.2.1)– (2.2.3) can be reduced to the problem for a system with D full row rank by applying the infinite zero compensation technique [5].

As in Chapter 2, we define $\mathbf{A}(\gamma)$ and $\bar{\mathbf{A}}(\gamma)$ as the sets of all $\mathbf{H}_\infty$ filters satisfying $\|T_{ed}\|_\infty < \gamma$ and $\|T_{ed}\|_\infty \leq \gamma$, respectively. It is shown in Theorem 2.1 that under the assumptions (A1) and (A2), the $\mathbf{H}_\infty$ filtering problem for the system (2.2.1)–(2.2.3) is solvable, namely $\mathbf{A}(\gamma) \neq \phi$, if and only if there exists a positive semi-definite stabilizing solution P to the ARE

$$P = APA^{\mathrm{T}} - (AP\widehat{C}^{\mathrm{T}} + \widehat{S})V^{-1}(AP\widehat{C}^{\mathrm{T}} + \widehat{S})^{\mathrm{T}} + BB^{\mathrm{T}} \tag{4.2.1a}$$

$$V = \begin{bmatrix} R + CPC^{\mathrm{T}} & CPL^{\mathrm{T}} \\ LPC^{\mathrm{T}} & -(\gamma^2 I_p - LPL^{\mathrm{T}}) \end{bmatrix} \tag{4.2.1b}$$

with $\widehat{V} = V_{21}V_{11}^{-1}V_{21}^{\mathrm{T}} - V_{22} > 0$, where $V = \begin{bmatrix} V_{11} & V_{21}^{\mathrm{T}} \\ V_{21} & V_{22} \end{bmatrix}$, and where $\widehat{C}$ and $\widehat{S}$ are defined by

$$\widehat{C} = \begin{bmatrix} C \\ L \end{bmatrix}, \quad \widehat{S} = \begin{bmatrix} BD^{\mathrm{T}} & 0 \end{bmatrix}$$

Under the assumption (A3), we define F and G by

$$F = A - BD^{\mathrm{T}}R^{-1}C, \quad G = B(I_m - D^{\mathrm{T}}R^{-1}D)$$

Under the assumptions (A1)–(A3), (C, F) is detectable and (F, G) has no uncontrollable modes on the unit circle. By the matrix inversion lemma, we see from (4.2.1) that

$$P = FPF^{\mathrm{T}} - FP\widehat{C}^{\mathrm{T}}V^{-1}\widehat{C}PF^{\mathrm{T}} + GG^{\mathrm{T}} \tag{4.2.2}$$

Let $\mathbf{P}(\gamma)$ be the set of all positive semi-definite solutions of the ARE (4.2.2), i.e. (4.2.1), satisfying $\widehat{V} > 0$. We also define $\gamma_{\mathrm{opt}} = \inf\{\gamma > 0 \,:\, \mathbf{A}(\gamma) \neq \phi\}$. It is clear from Theorem 2.1 that $\gamma > \gamma_{\mathrm{opt}}$ holds iff the ARE (4.2.1) has a stabilizing solution in $\mathbf{P}(\gamma)$.

Lemma 4.1: *Suppose that the ARE (4.2.2) has a stabilizing solution P in $\mathbf{P}(\gamma)$, namely $\gamma > \gamma_{\mathrm{opt}}$. Then,* Ker$P$ *coincides with the stable (F, G)-uncontrollable subspace.*

Proof: See Appendix 4.1. ■

We see from Lemma 4.1 that Ker P is independent of the parameter γ. We thus assume without loss of generality that F, G, C and L have the forms of

$$F = \begin{bmatrix} F_1 & F_{12} \\ 0 & F_2 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad L = \begin{bmatrix} L_1 & L_2 \end{bmatrix}$$

where (F_1, G_1) has no uncontrollable modes in the closed unit disk. Then, F_2 is stable since (F, G) has no uncontrollable modes on the unit circle. In the following, we also assume that F_1 is invertible.

Under the above assumptions, the stabilizing solution $P \in \mathbf{P}(\gamma)$ is of the form $P = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$ and P_1 is a unique positive definite stabilizing solution of

$$P_1 = F_1 P_1 F_1^{\mathrm{T}} - F_1 P_1 \widehat{C}_1^{\mathrm{T}} V^{-1} \widehat{C}_1 P_1 F_1^{\mathrm{T}} + G_1 G_1^{\mathrm{T}} \tag{4.2.3}$$

where $\widehat{C}_1 = \begin{bmatrix} C_1 \\ L_1 \end{bmatrix}$ and V is expressed as $V = \widehat{D} J_{mp} \widehat{D}^{\mathrm{T}} + \widehat{C}_1 P_1 \widehat{C}_1^{\mathrm{T}}$. Let $\mathbf{P}_1(\gamma)$ be the set of all positive definite solutions of the ARE (4.2.3) such that $\widehat{V} > 0$ holds. Then, from the above discussion, the existence of a stabilizing solution $P \in \mathbf{P}(\gamma)$ is equivalent to that of the stabilizing solution $P_1 \in \mathbf{P}_1(\gamma)$.

It is difficult to directly analyze the ARE (4.2.3) due to the indefinite coefficient matrix in V. Instead, we consider the ARE for P_1^{-1}, whose analysis is much easier than the ARE (4.2.3). By applying the matrix inversion lemma to the ARE (4.2.3), we observe that $X := P_1^{-1}$ is a unique anti-stabilizing solution of

$$\begin{aligned} X = {} & F_1^{\mathrm{T}} X F_1 + F_1^{\mathrm{T}} X G_1 (I_m - G_1^{\mathrm{T}} X G_1)^{-1} G_1^{\mathrm{T}} X F_1 \\ & - C_1^{\mathrm{T}} R^{-1} C_1 + \gamma^{-2} L_1^{\mathrm{T}} L_1 \end{aligned} \tag{4.2.4}$$

Since F_1 is assumed to be nonsingular, and since (4.2.3) is expressed as

$$P_1 = F_1 \bar{P}_1 F_1^{\mathrm{T}} + F_1 \bar{P}_1 L_1^{\mathrm{T}} \widehat{V}^{-1} L_1 \bar{P}_1 F_1^{\mathrm{T}} + G_1 G_1^{\mathrm{T}} \tag{4.2.5}$$

where $\bar{P}_1 = P_1 (I_r + C_1^{\mathrm{T}} R^{-1} C_1 P_1)^{-1}$ and $r = \mathrm{rank} P$, $P_1 \in \mathbf{P}_1(\gamma)$ implies that $X^{-1} - G_1 G_1^{\mathrm{T}} > 0$, i.e.

$$\tilde{V} := I_m - G_1^{\mathrm{T}} X G_1 > 0 \tag{4.2.6}$$

Similarly, it is easily proved that when there exists an anti-stabilizing solution $X > 0$ of the ARE (4.2.4) satisfying (4.2.6), $P_1 := X^{-1} > 0$ is a stabilizing solution of (4.2.3) in $\mathbf{P}_1(\gamma)$.

We give some results on the existence of a solution X to the ARE (4.2.4). To this end, we define $\mathbf{X}(\gamma)$ as the set of all solutions to the ARE (4.2.4) such that $\widetilde{V} > 0$.

Theorem 4.1: *For a given $\gamma > 0$, there exists an anti-stabilizing solution X in $\mathbf{X}(\gamma)$ if and only if*

$$\gamma > \gamma_X := \left\| \left[\begin{array}{c|c} \bar{F}_0 & \bar{G}_0 \\ \hline L_1 & 0 \end{array} \right] \right\|_\infty \tag{4.2.7}$$

where

$$\bar{F}_0 = F_1 + G_1(I_m - G_1^{\mathrm{T}} X_0 G_1)^{-1} G_1^{\mathrm{T}} X_0 F_1$$
$$\bar{G}_0 = G_1(I_m - G_1^{\mathrm{T}} X_0 G_1)^{-\frac{1}{2}}$$

and X_0 is an anti-stabilizing solution of the following ARE such that $\widetilde{V}_0 := I_m - G_1^{\mathrm{T}} X_0 G_1 > 0$.

$$X_0 = F_1^{\mathrm{T}} X_0 F_1 + F_1^{\mathrm{T}} X_0 G_1 (I_m - G_1^{\mathrm{T}} X_0 G_1)^{-1} G_1^{\mathrm{T}} X_0 F_1 - C_1^{\mathrm{T}} R^{-1} C_1 \tag{4.2.8}$$

Proof: Since (C_1, F_1) is detectable and since (F_1, G_1) has no uncontrollable modes in the closed unit disk, there exists a positive definite stabilizing solution to the ARE

$$P_0 = F_1 P_0 F_1^{\mathrm{T}} - F_1 P_0 C_1^{\mathrm{T}} (R + C_1 P_0 C_1^{\mathrm{T}})^{-1} C_1 P_0 F_1^{\mathrm{T}} + G_1 G_1^{\mathrm{T}} \tag{4.2.9}$$

We define $X_0 = P_0^{-1}$. It is easily verified by the matrix inversion lemma that X_0 is an anti-stabilizing solution of the ARE (4.2.8) with $\widetilde{V}_0 > 0$.

By simple but tedious calculations, we see that the solution of the ARE (4.2.4) is decomposed as $X = X_0 + M$, where M satisfies

$$M = \bar{F}_0^{\mathrm{T}} M \bar{F}_0 + \bar{F}_0^{\mathrm{T}} M \bar{G}_0 (I_m - \bar{G}_0^{\mathrm{T}} M \bar{G}_0)^{-1} \bar{G}_0^{\mathrm{T}} M \bar{F}_0 + \gamma^{-2} L_1^{\mathrm{T}} L_1 \tag{4.2.10}$$

Furthermore, we get

$$F_1 + G_1 \widetilde{V}^{-1} G_1^{\mathrm{T}} X F_1 = \bar{F}_0 + \bar{G}_0 (I_m - \bar{G}_0^{\mathrm{T}} M \bar{G}_0)^{-1} \bar{G}_0^{\mathrm{T}} M \bar{F}_0$$
$$\widetilde{V} = \widetilde{V}_0^{\frac{\mathrm{T}}{2}} (I_m - \bar{G}_0^{\mathrm{T}} M \bar{G}_0) \widetilde{V}_0^{\frac{1}{2}}$$

Thus, the existence of an anti-stabilizing solution X satisfying $\widetilde{V} > 0$ is equivalent to that of an anti-stabilizing solution M of the ARE (4.2.10) satisfying $I_m - \bar{G}_0^{\mathrm{T}} M \bar{G}_0 > 0$. By the bounded real lemma (Corollary 2.1), such a solution M exists if and only if $\gamma > \|L_1(\sigma I_r - \bar{F}_0)^{-1}\bar{G}_0\|_\infty$. This completes the proof. ■

It may be noted that the anti-stabilizing solution M is negative semi-definite since $\bar{F}_0$ is anti-stable and $I_m - \bar{G}_0^{\mathrm{T}} M \bar{G}_0 > 0$.

Moreover, γ_X is a lower bound of the parameter γ, for which the ARE (4.2.2) has a stabilizing solution, because X^{-1} is a stabilization of (4.2.3) if the inverse of X exists.

Theorem 4.2: *Suppose that $\mathbf{X}(\gamma) \neq \phi$ for a given $\gamma > 0$. Then, there exists a matrix $X \in \mathbf{X}(\gamma)$ such that*

$$X \geq X_{\mathrm{a}}, \quad \forall X_{\mathrm{a}} \in \mathbf{X}(\gamma)$$

and it is an anti-strong solution, i.e. all the eigenvalues of $\widehat{F} := F_1 + G_1 \widetilde{V}^{-1} G_1^{\mathrm{T}} X F_1$ do not belong to the open unit disk.

Proof: See Appendix 4.2. ■

Corollary 4.1: *For a given $\gamma > 0$, suppose that $\mathbf{P}_1(\gamma) \neq \phi$ holds. Then there exists a matrix $P_1 \in \mathbf{P}_1(\gamma)$ such that*

$$P_1 \leq P_{\mathrm{a}}, \quad \forall P_{\mathrm{a}} \in \mathbf{P}_1(\gamma)$$

and it is a strong solution, i.e. all the eigenvalues of $F_1 - F_1 P_1 \widehat{C}_1^{\mathrm{T}} V^{-1} \widehat{C}_1$ belong to the closed unit disk rather than the open unit disk.

Theorem 4.3 shows the monotonicity and concavity of the anti-stabilizing solution X of (4.2.4).

Theorem 4.3: *For given positive constants γ_1 and γ_2, suppose that the ARE (4.2.4) has anti-stabilizing solutions $X^{(i)}$ in $\mathbf{X}(\gamma_i)$ $(i = 1, 2)$.*
(a) If $\gamma_1 > \gamma_2 \, (> \gamma_X)$ holds, then $X^{(1)} \geq X^{(2)}$ holds.

(b) Define $\gamma_0 = \alpha\gamma_1 + \beta\gamma_2$ with $\alpha + \beta = 1,\ \alpha,\ \beta \geq 0$. Then, there exists an anti-stabilizing solution $X^{(0)}$ in $\mathbf{X}(\gamma_0)$, and we have

$$X^{(0)} \geq \alpha X^{(1)} + \beta X^{(2)}$$

Proof: (a) The AREs corresponding to γ_i $(i = 1, 2)$ are given by

$$\begin{aligned} X^{(i)} &= F_1^{\mathrm{T}} X^{(i)} F_1 + F_1^{\mathrm{T}} X^{(i)} G_1 \widetilde{V}_i^{-1} G_1^{\mathrm{T}} X^{(i)} F_1 \\ &\quad - C_1^{\mathrm{T}} R^{-1} C_1 + \gamma_i^{-2} L_1^{\mathrm{T}} L_1 \end{aligned} \tag{4.2.11}$$

where $\widetilde{V}_i = I_m - G_1^{\mathrm{T}} X^{(i)} G_1$. We define

$$\Gamma_i = \widetilde{V}_i^{-1} G_1^{\mathrm{T}} X^{(i)} F_1, \quad \widehat{F}_i = F_1 + G_1 \Gamma_i$$

Then (4.2.11) is rewritten as

$$\begin{aligned} X^{(i)} &= \widehat{F}_1^{\mathrm{T}} X^{(i)} \widehat{F}_1 - \Gamma_1^{\mathrm{T}} \Gamma_1 - C_1^{\mathrm{T}} R^{-1} C_1 + \gamma_i^{-2} L_1^{\mathrm{T}} L_1 \\ &\quad + (\Gamma_i - \Gamma_1)^{\mathrm{T}} \widetilde{V}_2 (\Gamma_i - \Gamma_1) \qquad (i = 1, 2) \end{aligned} \tag{4.2.12}$$

Hence we obtain

$$\begin{aligned} X^{(1)} - X^{(2)} &= \widehat{F}_1^{\mathrm{T}} (X^{(1)} - X^{(2)}) \widehat{F}_1 - (\gamma_2^{-2} - \gamma_1^{-2}) L_1^{\mathrm{T}} L_1 \\ &\quad - (\Gamma_2 - \Gamma_1)^{\mathrm{T}} \widetilde{V}_2 (\Gamma_2 - \Gamma_1) \end{aligned}$$

Since $\widehat{F}_1$ is anti-stable and $\widetilde{V}_2 > 0$, by Lyapunov's theorem, $X^{(1)} - X^{(2)} \geq 0$ holds when $\gamma_1 > \gamma_2$.

(b) By the definition of γ_0, we see that $\gamma_0 \geq \min\{\gamma_1, \gamma_2\} > \gamma_X$, and hence the ARE (4.2.4) has an anti-stabilizing solution $X^{(0)}$ in $\mathbf{X}(\gamma_0)$. The AREs for γ_i are expressed as

$$\begin{aligned} X^{(i)} &= \widehat{F}_0^{\mathrm{T}} X^{(i)} \widehat{F}_0 - \Gamma_0^{\mathrm{T}} \Gamma_0 - C_1^{\mathrm{T}} R^{-1} C_1 + \gamma_i^{-2} L_1^{\mathrm{T}} L_1 \\ &\quad + (\Gamma_0 - \Gamma_i)^{\mathrm{T}} \widetilde{V}_i (\Gamma_0 - \Gamma_i) \qquad (i = 0, 1, 2) \end{aligned}$$

We now define $\widehat{X} = \alpha X^{(1)} + \beta X^{(2)} - X^{(0)}$. Then, we see from the above equations that

$$\begin{aligned} \widehat{X} - \widehat{F}_0^{\mathrm{T}} \widehat{X} \widehat{F}_0 &= \alpha (\Gamma_0 - \Gamma_1)^{\mathrm{T}} \widetilde{V}_1 (\Gamma_0 - \Gamma_1) + \beta (\Gamma_0 - \Gamma_2)^{\mathrm{T}} \widetilde{V}_2 (\Gamma_0 - \Gamma_2) \\ &\quad + (\alpha \gamma_1^{-2} + \beta \gamma_2^{-2} - \gamma_0^{-2}) L_1^{\mathrm{T}} L_1 \end{aligned}$$

Since $\alpha\gamma_1^{-2}+\beta\gamma_2^{-2} > \gamma_0^{-2}$ holds for any $\alpha,\ \beta \geq 0$ such that $\alpha+\beta = 1$, the right-hand side of the above equation is positive semi-definite. Noting that $\widehat{F}_0$ is anti-stable, we conclude that $\widehat{X} \leq 0$ by Lyapunov's theorem. This completes the proof of (b). ■

Corollary 4.2: *For given positive constants γ_1 and γ_2, suppose that the ARE (4.2.2), equivalently (4.2.1), has stabilizing solutions $P^{(i)}$ in $\mathbf{P}(\gamma_i)$ $(i = 1,2)$.*

(a) If $\gamma_1 > \gamma_2, (> \gamma_{\rm opt})$ holds, then $P^{(1)} \leq P^{(2)}$ and $\bar{P}^{(1)} \leq \bar{P}^{(2)}$ hold, where $\bar{P}^{(i)} = P^{(i)}(I_n + C_1^{\rm T}R^{-1}C_1P^{(i)})^{-1}$.

(b) Define $\gamma_0 = \alpha\gamma_1 + \beta\gamma_2$ with $\alpha+\beta = 1,\ \alpha,\ \beta \geq 0$. Then, there exists a stabilizing solution $P^{(0)}$ in $\mathbf{P}(\gamma_0)$, and we have

$$P^{(0)} \leq \alpha P^{(1)} + \beta P^{(2)}$$

Proof: The inequality $P^{(1)} \leq P^{(2)}$ and the part (b) are immediate from Theorem 4.3. The inequality $\bar{P}^{(1)} \leq \bar{P}^{(2)}$ follows from the fact that

$$\begin{aligned}\bar{P}^{(2)} - \bar{P}^{(1)} = &(I_n - K^{(2)}C)(P^{(2)} - P^{(1)})(I_n - K^{(2)}C)^{\rm T}\\ &+(K^{(1)} - K^{(2)})(R + CP^{(1)}C^{\rm T})(K^{(1)} - K^{(2)})^{\rm T}\end{aligned}$$

where $K^{(i)} = P^{(i)}C^{\rm T}(R + CP^{(i)}C^{\rm T})^{-1}$. ■

Theorem 4.4: *The anti-stabilizing solution $X \in \mathbf{X}(\gamma)$ converges to a finite anti-strong solution in $\mathbf{X}(\gamma_X)$ as γ tends to $\gamma_X + 0$.*

Proof: Since $\gamma_X = \|L_1(\sigma I_r - \bar{F}_0)^{-1}\bar{G}_0\|_\infty$, we see from Theorem 2.1 of [6] that there exists a negative semi-definite anti-strong solution to the ARE (4.2.10) such that $I_m - \bar{G}_0^{\rm T}M\bar{G}_0 > 0$ at $\gamma = \gamma_X$. Thus, there exists an anti-strong solution to the ARE (4.2.4) in $\mathbf{X}(\gamma_X)$. Let X' be such an anti-strong solution, which is one of the maximal elements of $\mathbf{X}(\gamma_X)$ by Theorem 4.2. It is also easy to verify that any maximal element of $\mathbf{X}(\gamma)$ is anti-strong, if $\mathbf{X}(\gamma) \neq \phi$. Moreover, similarly to the proof of Theorem 4.3 (a), we can easily prove that the anti-stabilizing solution X in $\mathbf{X}(\gamma)$ satisfies $X \geq X'$ for any γ larger than γ_X. Thus, the anti-stabilizing solution X in $\mathbf{X}(\gamma)$ is bounded below and monotonically non-decreasing with respect to $\gamma\,(> \gamma_X)$. Therefore, the anti-stabilizing solution converges to a finite anti-strong solution in $\mathbf{X}(\gamma_X)$ as γ tends to $\gamma_X + 0$. ■

Theorem 4.3 shows that the eigenvalues of the anti-stabilizing solution $X \in \mathbf{X}(\gamma)$ are the non-decreasing concave functions of γ. Similarly, by Corollary 4.2, the eigenvalues of the stabilizing solution $P_1 \in \mathbf{P}_1(\gamma)$ are the non-increasing convex functions of γ.

By Theorems 4.1, 4.3 and Corollary 4.2, we see that there exists a stabilizing solution $P \in \mathbf{P}(\gamma)$ if $\gamma > \gamma_X$ and $\det(X) \neq 0$, so that $\gamma_{\mathrm{opt}} \geq \gamma_X$. The behavior of the stabilizing solution P near γ_{opt} depends on the eigenvalues of the anti-stabilizing solution X of (4.2.4) as $\gamma \to \gamma_X + 0$. There are two possibilities for the behavior of X near $\gamma = \gamma_X$.

Case 1 ($\lim_{\gamma \downarrow \gamma_X} \lambda_{\min}(X) > 0$): In this case, X converges to a finite positive definite solution as γ tends to $\gamma_{\mathrm{opt}} + 0$. Hence, the stabilizing solution P converges to a finite strong solution. In this case, $\gamma_{\mathrm{opt}} = \gamma_X$ holds.

Especially, in the case where $L_1 = 0$ holds, a positive definite anti-stabilizing solution of (4.2.4) always exists independently of γ. Thus, we see that $\gamma_{\mathrm{opt}} = \gamma_X = 0$ when $L_1 = 0$.

Case 2 ($\lim_{\gamma \downarrow \gamma_X} \lambda_{\min}(X) \leq 0$): From Theorem 4.3, we see that as γ decreases, an eigenvalue of X crosses zero to change its sign from positive to negative. Thus, in this case, there exists a point $\gamma\,(\geq \gamma_X)$ such that X becomes singular. At this point, the stabilizing solution P of the ARE (4.2.2) diverges to infinity since $P = \mathrm{diag}[\,X^{-1} \;\; 0\,]$. Moreover, we see from Corollary 4.2 that sign changes of the eigenvalues of P do not result from zero-crossing. As γ decreases, the eigenvalues of P change their signs by escaping to $+\infty$ and reappearing from $-\infty$.

Lemma 4.2: *Suppose that* $\mathrm{Ker}\, C_1 \cap \mathrm{Ker} L_1 = 0$ *and/or* $\mathrm{Ker}\, C_1$ *is* F_1*-invariant. We also define* $\bar{P} = P(I_n + C^{\mathrm{T}} R^{-1} C P)^{-1}$ *for the stabilizing solution* P *of (4.2.1). Then,* $\bar{P}$ *is finite at* $\gamma = \gamma_{\mathrm{opt}}$.

Proof: We see from the definition that

$$\bar{P} = \mathrm{diag}[\,\bar{P}_1 \;\; 0\,] = \mathrm{diag}[\,(X + C_1^{\mathrm{T}} R^{-1} C_1)^{-1} \;\; 0\,]$$

Hence, we need to show that $X + C_1^{\mathrm{T}} R^{-1} C_1 > 0$ for the anti-stabilizing solution X in $\mathbf{X}(\gamma_{\mathrm{opt}})$.

We assume on the contrary that $X + C_1^{\mathrm{T}} R^{-1} C_1$ is singular. Note that X is positive semi-definite and $\widetilde{V} > 0$ holds at $\gamma = \gamma_{\mathrm{opt}}$. Let ξ be a nonzero element of $\mathrm{Ker}(X + C_1^{\mathrm{T}} R^{-1} C_1)$.

Since $X \geq 0$ and $R > 0$, $\xi \in \mathrm{Ker}(X + C_1^{\mathrm{T}} R^{-1} C_1)$ implies $X\xi = 0$ and $C_1\xi = 0$. Pre-multiplying (4.2.4) by ξ^{H} and post-multiplying by ξ yield

$$\xi^{\mathrm{H}}(F_1^{\mathrm{T}} X F_1 + F_1^{\mathrm{T}} X G_1 \widetilde{V}^{-1} G_1^{\mathrm{T}} X F_1 + \gamma^{-2} L_1^{\mathrm{T}} L_1)\xi = 0$$

It follows from $X \geq 0$ and $\widetilde{V} > 0$ that $XF_1\xi = 0$ and $L_1\xi = 0$ hold.

In the case where $\mathrm{Ker} C_1 \cap \mathrm{Ker} L_1 = 0$, we get $\xi = 0$ from $L_1\xi = 0$ and $C_1\xi = 0$, a contradiction.

Otherwise, we assume that $\mathrm{Ker} C_1$ is F_1-invariant. Then, we see from $XF_1\xi = 0$ that

$$\widehat{F}\xi = F_1\xi, \quad \forall \xi \in \mathrm{Ker}(X + C_1^{\mathrm{T}} R^{-1} C_1) \tag{4.2.13}$$

where $\widehat{F} = F_1 + G_1 \widetilde{V}^{-1} G_1^{\mathrm{T}} X F_1$. Thus, $\mathrm{Ker}(X + C_1^{\mathrm{T}} R^{-1} C_1)$ is invariant under $\widehat{F}$. Since $\widehat{F}$ is anti-stable, $\widehat{F}$ restricted to $\mathrm{Ker}(X + C_1^{\mathrm{T}} R^{-1} C_1)$ has an unstable eigenvalue λ and a corresponding eigenvector x:

$$\widehat{F}x = \lambda x, \quad |\lambda| > 1, \ x \in \mathrm{Ker}(X + C_1^{\mathrm{T}} R^{-1} C_1)$$

It follows from (4.2.13) that $F_1 x = \lambda x$. This contradicts the detectability of (C_1, F_1).

Consequently, it has been proved that $X + C_1^{\mathrm{T}} R^{-1} C_1$ is nonsingular. Hence, $\bar{P}$ is finite at $\gamma = \gamma_{\mathrm{opt}}$. ■

This theorem shows that there is a possibility that $\bar{P}$ remains finite even though $P = \mathrm{diag}[\, X^{-1} \ 0 \,]$ diverges to infinity. For example, if L is nonsingular, $\bar{P}$ has a finite value at the optimum.

3. Parametrization of All $\mathbf{H}_\infty$ Filters

In this section, we examine the behavior of the size of $\bar{\mathbf{A}}(\gamma)$ for the variation of γ based upon the results given in the previous section. For this purpose, we consider the parametrization of all $\mathbf{H}_\infty$ filters $T_{\mathrm{f}}(\sigma) \in \bar{\mathbf{A}}(\gamma)$.

We hereafter assume that $\gamma > \gamma_{\mathrm{opt}}$ holds. Then, the conditions (a) and (b) of Theorem 2.1 hold, and by Theorem 2.2, the $\mathbf{H}_\infty$ filter $T_{\mathrm{f}}(\sigma) \in \bar{\mathbf{A}}(\gamma)$ is given by

$$T_{\mathrm{f}}(\sigma) = -(U\Omega_{12} - \Omega_{22})^{-1}(U\Omega_{11} - \Omega_{12}), \quad U(\sigma) \in \overline{\mathbf{BH}}_{\infty}^{p\times q}$$

$$\Omega(\sigma) = W^{-1}\left[\begin{array}{c|cc} A_K & -K_1 & -K_2 \\ \hline C & I_q & 0 \\ L & 0 & I_p \end{array}\right]$$

where W, $K = [K_1\ K_2]$ and A_K are defined by (2.4.13),(2.4.16) and (2.4.17), respectively. Since a matrix W satisfying $WJ_{qp}W^{\mathrm{T}} = V$ is not unique, the degree of freedom of $T_{\mathrm{f}}(\sigma)$ is expressed in terms of two parameters $U(\sigma)$ and W. This observation makes it difficult to evaluate the size of $\bar{\mathbf{A}}(\gamma)$ using the above parametrization. Therefore, we first derive a new parametrization where the degree of freedom is condensed into only one free parameter.

Theorem 4.5: *Suppose that $\mathbf{A}(\gamma)$ is not empty. Then $T_{\mathrm{f}}(\sigma) \in \bar{\mathbf{A}}(\gamma)$ is parametrized by*

$$T_{\mathrm{f}}(\sigma) = -(\widehat{\Omega}_{22} - Z\widehat{\Omega}_{12})^{-1}(\widehat{\Omega}_{21} - Z\widehat{\Omega}_{11}) \tag{4.3.1}$$

$$\widehat{\Omega}(\sigma) = \begin{bmatrix} \widehat{\Omega}_{11} & \widehat{\Omega}_{12} \\ \widehat{\Omega}_{21} & \widehat{\Omega}_{22} \end{bmatrix} = \begin{bmatrix} I_q & 0 \\ -V_{21}V_{11}^{-1} & I_p \end{bmatrix} \left[\begin{array}{c|cc} A_K & -K_1 & -K_2 \\ \hline C & I_q & 0 \\ L & 0 & I_p \end{array}\right] \tag{4.3.2}$$

where $Z(\sigma)$ an arbitrary transfer matrix in $\mathbf{RH}_{\infty}^{p\times q}$ such that

$$Z(\sigma)V_{11}Z^{\sim}(\sigma) \le \widehat{V} \tag{4.3.3}$$

Proof: We define

$$Z(\sigma) = \widehat{Z}(\sigma) - V_{21}V_{11}^{-1} \tag{4.3.4}$$

$$\widehat{Z}(\sigma) = -\widehat{Z}_2^{-1}(\sigma)\widehat{Z}_1(\sigma), \quad \begin{bmatrix} \widehat{Z}_1 & \widehat{Z}_2 \end{bmatrix} = \begin{bmatrix} -U & I_p \end{bmatrix} W^{-1} \tag{4.3.5}$$

Then, simple algebraic manipulation yields (4.3.1) and (4.3.2). We also obtain

$$\begin{aligned} UU^{\sim} - \gamma^2 I_p &= \begin{bmatrix} \widehat{Z}_1 & \widehat{Z}_2 \end{bmatrix} WJ_{qp}W^{\mathrm{T}} \begin{bmatrix} \widehat{Z}_1^{\sim} \\ \widehat{Z}_2^{\sim} \end{bmatrix} \\ &= \widehat{Z}_2(ZV_{11}Z^{\sim} - \widehat{V})\widehat{Z}_2^{\sim} \end{aligned} \tag{4.3.6}$$

Note that $[\,\widehat{Z}_1(\sigma)\ \ \widehat{Z}_2(\sigma)\,]$ is stable and has a right inverse in $\mathbf{RL}_{\infty}$ since $U(\sigma) \in \mathbf{RH}_{\infty}^{p\times q}$. We here assume that $\widehat{Z}_2(\sigma)$ has a zero λ such that $|\lambda| \ge 1$. Then, there exists a non-zero vector ξ such that $\xi^{\mathrm{H}}\widehat{Z}_2(\lambda) = 0$. Thus, we get

$$\xi^{\mathrm{H}}(UU^{\sim} - \gamma^2 I_p)\xi = \xi^{\mathrm{H}}\widehat{Z}_1 V_{11}\widehat{Z}_1^{\sim}\xi, \qquad \sigma = \lambda$$

Since $\|U\|_\infty \leq \gamma$ and $V_{11} > 0$, $\xi^{\mathrm{H}} \widehat{Z}_1(\lambda) = 0$ holds. This contradicts the right invertibility of $[\,\widehat{Z}_1 \;\; \widehat{Z}_2\,]$. Thus, $\widehat{Z}_2(\sigma)$ is unimodular, so that $Z(\sigma)$ is stable. Since $\widehat{Z}_2(\sigma)$ is unimodular, $\|U\|_\infty \leq \gamma$ implies $Z(\sigma)V_{11}Z^{\sim}(\sigma) \leq \widehat{V}$.

It is also shown from (4.3.4) and (4.3.5) that

$$U(\sigma) = -(W_{22} - \widehat{Z}W_{12})^{-1}(W_{21} - \widehat{Z}W_{11})$$

Hence, similarly to the above discussion, we can show that $U(\sigma)$ belongs to $\overline{\mathbf{B}}\mathbf{H}_\infty(\gamma)$ if $Z(\sigma)$ is stable and satisfies (4.3.3). ■

If we fix the matrix W, then the mapping from $U(\sigma)$ to $Z(\sigma)$ is bijective. It may be also noted that $U(\sigma) = 0 \Leftrightarrow Z(\sigma) = 0$ holds when W is given by

$$W = \begin{bmatrix} V_{11}^{\frac{1}{2}} & 0 \\ V_{21}V_{11}^{-\frac{1}{2}} & -(V_{21}V_{11}^{-1}V_{21}^{\mathrm{T}} - V_{22})^{\frac{1}{2}} \end{bmatrix}$$

Moreover, taking $Z(\sigma) = 0$ yields the central filter defined by (2.4.31):

$$T_{\mathrm{f}}(\sigma) = \left[\begin{array}{c|c} A - K_\infty C & K_\infty \\ \hline L - M_\infty C & M_\infty \end{array}\right]$$

where

$$K_\infty = (APC^{\mathrm{T}} + BD^{\mathrm{T}})(R + CPC^{\mathrm{T}})^{-1}$$
$$M_\infty = LPC^{\mathrm{T}}(R + CPC^{\mathrm{T}})^{-1}$$

Furthermore, as shown in Lemma 4.2, $\bar{P}$ is finite at the optimum γ_{opt} under a certain condition. For the central $\mathbf{H}_\infty$ filter of (2.4.31), K_∞ and M_∞ can be expressed as

$$K_\infty = (F\bar{P}C^{\mathrm{T}} + BD^{\mathrm{T}})R^{-1}, \quad M_\infty = L\bar{P}C^{\mathrm{T}}R^{-1}$$

Therefore, there is a possibility that the central $\mathbf{H}_\infty$ filter (2.4.31) with finite coefficients exists even though P diverges to infinity at the optimum γ_{opt}.

We define

$$\mathbf{Z}(\gamma) = \{Z(\sigma) |\; ZV_{11}Z^{\sim} \leq \widehat{V}, \;\; Z(\sigma) \in \mathbf{RH}_\infty^{p\times q}\}$$

Then, $\mathbf{Z}(\gamma)$ is a bounded closed convex set, and the following inequality holds for all $Z(\sigma) \in \mathbf{Z}(\gamma)$.

$$\|Z\|_\infty \leq \sqrt{\frac{\lambda_{\max}(\widehat{V})}{\lambda_{\min}(V_{11})}} \tag{4.3.7}$$

Since $\widehat{\Omega}(\sigma)$ is uniquely determined by γ, the degree of freedom contained in the $\mathbf{H}_\infty$ filter is condensed into the free parameter $Z(\sigma)$. Therefore, the size of the solution set $\bar{\mathbf{A}}(\gamma)$ is identical to that of $\mathbf{Z}(\gamma)$. Note that the quantity on the right-hand side of (4.3.7) is useful as a measure of the size of $\mathbf{Z}(\gamma)$.

Since $\mathbf{Z}(\gamma)$ is characterized by V_{11} and $\widehat{V}$ which depend on γ and P, the results given in the previous section are very useful to understand the behavior of the set $\mathbf{Z}(\gamma)$ as γ changes.

Theorem 4.6: *The size of the set $\mathbf{Z}(\gamma)$ is monotonically increasing with respect to γ in the sense that*

$$\gamma_{\mathrm{opt}} < \gamma_2 < \gamma_1 \Longrightarrow \mathbf{Z}(\gamma_2) \subset \mathbf{Z}(\gamma_1)$$

Proof: Let $V_{11}^{(i)}$ and $\widehat{V}^{(i)}$ denote the values of V_{11} and $\widehat{V}$ for γ_i $(i = 1, 2)$, respectively.

Since V_{11} and $\widehat{V}$ are respectively expressed as $V_{11} = R + CPC^{\mathrm{T}}$ and $\widehat{V} = \gamma^2 I_p - L\bar{P}L^{\mathrm{T}}$, it follows from Corollary 4.2 that

$$V_{11}^{(1)} \leq V_{11}^{(2)}, \quad \widehat{V}^{(1)} > \widehat{V}^{(2)} \tag{4.3.8}$$

We assume that $Z(\sigma)$ belongs to $\mathbf{Z}(\gamma_2)$, i.e. $ZV_{11}^{(2)}Z^{\sim} \leq \widehat{V}^{(2)}$. Then, we see from (4.3.8) that

$$ZV_{11}^{(1)}Z^{\sim} \leq ZV_{11}^{(2)}Z^{\sim} \leq \widehat{V}^{(2)} < \widehat{V}^{(1)}$$

Thus, $Z(\sigma) \in \mathbf{Z}(\gamma_1)$ holds, and hence $\mathbf{Z}(\gamma_2) \subseteq \mathbf{Z}(\gamma_1)$.

Furthermore, it is easy to verify that there exists a constant matrix Z in $\mathbf{Z}(\gamma_1)$ such that $\|Z\| = \sqrt{\lambda_{\max}(\widehat{V}^{(1)})/\lambda_{\min}(V_{11}^{(1)})}$. In this case, there exists a nonzero vector $\xi \in \mathbf{C}^p$ satisfying

$$\xi^{\mathrm{H}} Z V_{11}^{(1)} Z^{\mathrm{T}} \xi = \xi^{\mathrm{H}} \widehat{V}^{(1)} \xi \tag{4.3.9}$$

Hence, we obtain

$$\xi^{\mathrm{H}} Z V_{11}^{(2)} Z^{\mathrm{T}} \xi \geq \xi^{\mathrm{H}} Z V_{11}^{(1)} Z^{\mathrm{T}} \xi = \xi^{\mathrm{H}} \widehat{V}^{(1)} \xi > \xi^{\mathrm{H}} \widehat{V}^{(2)} \xi$$

This implies that such a $Z \in \mathbf{Z}(\gamma_1)$ does not belong to $\mathbf{Z}(\gamma_2)$. Consequently, we have proved $\mathbf{Z}(\gamma_2) \subset \mathbf{Z}(\gamma_1)$. ■

In the following, we consider the size of the set $\bar{\mathbf{A}}(\gamma)$ as γ tends to the optimum γ_{opt}. When $\lim_{\gamma \downarrow \gamma_X} \lambda_{\min}(X) > 0$, all optimal $\mathbf{H}_\infty$ filters are parametrized by Theorem 4.5 because the ARE (4.2.1), i.e. (4.2.2), has a finite strong solution with $\widehat{V} > 0$ at the optimum γ_{opt}. However, if $\lim_{\gamma \downarrow \gamma_X} \lambda_{\min}(X) \leq 0$ holds, the largest eigenvalue of P diverges to $+\infty$ as γ tends to γ_{opt} as observed in the previous section. Hence, it is impossible to characterize the set $\bar{\mathbf{A}}(\gamma_{\mathrm{opt}})$ in terms of P. Hereafter, we wish to study the limit of the set $\mathbf{Z}(\gamma)$ as γ goes to $\gamma_{\mathrm{opt}} + 0$ under the assumption that $\lim_{\gamma \downarrow \gamma_X} \lambda_{\min}(X) \leq 0$.

Since V_{11} and $\widehat{V}$ are symmetric, there exist orthogonal matrices $E(\gamma)$ and $\widehat{E}(\gamma)$ such that

$$V_{11} = E^{\mathrm{T}} \begin{bmatrix} \Lambda' & 0 \\ 0 & \Lambda \end{bmatrix} E, \quad \widehat{V} = \widehat{E}^{\mathrm{T}} \begin{bmatrix} M & 0 \\ 0 & M' \end{bmatrix} \widehat{E} \tag{4.3.10}$$

where Λ, Λ', M and M' are the diagonal matrices satisfying

$$\begin{aligned} \Lambda_o &:= \lim_{\gamma \downarrow \gamma_{\mathrm{opt}}} \Lambda < +\infty, \qquad \lim_{\gamma \downarrow \gamma_{\mathrm{opt}}} \Lambda' = +\infty I_{q-h} \\ M_o &:= \lim_{\gamma \downarrow \gamma_{\mathrm{opt}}} M > 0, \qquad \lim_{\gamma \downarrow \gamma_{\mathrm{opt}}} M' = 0 \end{aligned}$$

and $f\,(< p)$ and $h\,(< q)$ denote the dimensions of M and Λ, respectively. Further, we define $\widehat{E}ZE^{\mathrm{T}} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$ for $Z(\sigma) \in \mathbf{Z}(\gamma)$ according to (4.3.10). We see from (4.3.3) that

$$\begin{aligned} Y_{11}\Lambda' Y_{11}^{\sim} + Y_{12}\Lambda Y_{12}^{\sim} &\leq M \\ Y_{21}\Lambda' Y_{21}^{\sim} + Y_{22}\Lambda Y_{22}^{\sim} &\leq M' \end{aligned}$$

Since $\Lambda' \to \infty I_{q-h}$ and $M' \to 0$ when γ tends to $\gamma_{\mathrm{opt}} + 0$, Y_{11}, Y_{21} and Y_{22} converge to 0. Then the remaining free parameter $Y_{12}(\sigma)$ satisfies

$$Y_{12}(\sigma)\Lambda_o Y_{12}^{\sim}(\sigma) \leq M_o$$

Especially, if $f = 0$ holds, then we get $\widehat{E}ZE^{\mathrm{T}} = [Y_{21}\ Y_{22}]$. Also, $h = 0$ implies $\widehat{E}ZE^{\mathrm{T}} = \begin{bmatrix} Y_{11} \\ Y_{21} \end{bmatrix}$. It thus follows from (4.3.3) that $\lim_{\gamma \downarrow \gamma_{\mathrm{opt}}} \mathbf{Z}(\gamma) = 0$ holds in these cases.

The next theorem summarizes the above results.

Theorem 4.7: *Suppose that* $\lim_{\gamma\downarrow\gamma_X} \lambda_{\min}(X) \le 0$ *holds.*
(i) If $f > 0$ *and* $h > 0$*, then we have* $\lim_{\gamma\downarrow\gamma_{\rm opt}} \mathbf{Z}(\gamma) = \mathbf{Z}_{\rm o}$, *where*

$$\mathbf{Z}_{\rm o} = \{Z(\sigma)|\ Z = \widehat{E}_{\rm o}^{\rm T} \begin{bmatrix} 0 & Y_{12} \\ 0 & 0 \end{bmatrix} E_{\rm o},\ \ Y_{12}\Lambda_{\rm o}Y_{12}^{\sim} \le M_{\rm o},$$

$$Y_{12}(\sigma) \in \mathbf{RH}_\infty^{f\times h}\}$$

$$E_{\rm o} = \lim_{\gamma\downarrow\gamma_{\rm opt}} E(\gamma), \qquad \widehat{E}_{\rm o} = \lim_{\gamma\downarrow\gamma_{\rm opt}} \widehat{E}(\gamma)$$

(ii) If $f = 0$ *and/or* $h = 0$ *holds, then we have* $\lim_{\gamma\downarrow\gamma_{\rm opt}} \mathbf{Z}(\gamma) = 0$.

The above theorem shows that, in the case where $\lim_{\gamma\downarrow\gamma_X} \lambda_{\min}(X) \le 0$, the degree of freedom of the $\mathbf{H}_\infty$ filter decreases at the optimum $\gamma_{\rm opt}$.

It may be noted that the case (ii) in the above theorem is not a rare case. In fact, the problem which has a scalar measurement $y_k \in \mathbf{R}$, namely $q = 1$ and $h = 0$, is an example of the case (ii), which can be seen in many applications.

4. $\mathbf{H}_2/\mathbf{H}_\infty$ Filtering Problem

In this section, we propose an $\mathbf{H}_2/\mathbf{H}_\infty$ filtering algorithm which makes use of the free parameter $Z(\sigma) \in \mathbf{Z}(\gamma)$. Suppose that $\mathbf{A}(\gamma)$ is not empty for a given $\gamma > 0$. The $\mathbf{H}_2/\mathbf{H}_\infty$ filtering problem is the optimization problem defined by

$$\min_{T_{\rm f}(\sigma)\in\bar{\mathbf{A}}(\gamma)} \|T_{ed}\|_2, \quad \text{or equivalently,} \quad \min_{Z(\sigma)\in\mathbf{Z}(\gamma)} \|T_{ed}\|_2$$

In the following, we assume for simplicity that the free parameter $Z(\sigma)$ is a constant matrix. In this case, we get

$$T_{\rm f}(\sigma) = \left[\begin{array}{c|c} A_\infty + K_2ZC & K_\infty - K_2Z \\ \hline L_\infty + ZC & M_\infty - Z \end{array}\right] \tag{4.4.1}$$

$$T_{ed}(\sigma) = \left[\begin{array}{c|c} A_\infty + K_2ZC & B_\infty + K_2ZD \\ \hline L_\infty + ZC & (Z - M_\infty)D \end{array}\right] \tag{4.4.2}$$

Hence, the $\mathbf{H}_2$ norm of $T_{ed}(\sigma)$ is given by

$$\|T_{ed}\|_2^2 = \mathrm{Tr}\{(L_\infty + ZC)Y(L_\infty + ZC)^{\rm T} + (Z - M_\infty)R(Z - M_\infty)^{\rm T}\} \tag{4.4.3}$$

where Y is the unique positive semi-definite solution to the Lyapunov equation:

$$\begin{aligned}0 = -Y + (A_\infty + K_2 Z C) Y (A_\infty + K_2 Z C)^{\mathrm{T}} \\ +(B_\infty + K_2 Z C)(B_\infty + K_2 Z C)^{\mathrm{T}}\end{aligned} \tag{4.4.4}$$

Let $J_2(Z,Y)$ and $\mathrm{Lyap}(Z,Y)$ be the right-hand sides of (4.4.3) and (4.4.4), respectively. It then follows that the simplified $\mathbf{H}_2/\mathbf{H}_\infty$ filtering problem is formulated as the optimization problem:

$$\min\{J_2(Z,Y) \;:\; Z \in \mathbf{Z}_{\mathrm{const}}(\gamma),\ \mathrm{Lyap}(Z,Y) = 0\} \tag{4.4.5}$$

where $\mathbf{Z}_{\mathrm{const}}(\gamma)$ is the subset of $\mathbf{Z}(\gamma)$ defined by

$$\mathbf{Z}_{\mathrm{const}}(\gamma) = \{Z \mid Z \in \mathbf{R}^{p\times q},\ \ Z V_{11} Z^{\mathrm{T}} \le \widehat{V}\}$$

A necessary condition for the existence of a solution to this optimization problem is given by the following theorem.

Theorem 4.8: *Suppose that $\gamma > \gamma_{\mathrm{opt}}$ holds, and that $T_{\mathrm{f}}(\sigma) \in \mathbf{A}(\gamma)$ is given by (4.4.1) with Z in $\mathbf{Z}_{\mathrm{const}}(\gamma)$. If Z is a solution to the optimization problem of (4.4.5, then there exist positive semi-definite matrices Λ and Y such that*

$$Y = (A_\infty + K_2 Z C) Y (A_\infty + K_2 Z C)^{\mathrm{T}} + (B_\infty + K_2 Z D)(B_\infty + K_2 Z D)^{\mathrm{T}} \tag{4.4.6}$$

$$\Lambda = (A_\infty + K_2 Z C)^{\mathrm{T}} \Lambda (A_\infty + K_2 Z C) + (L_\infty + Z C)^{\mathrm{T}} (L_\infty + Z C) \tag{4.4.7}$$

$$Z = \Xi^{-1}\{(M_\infty - \widetilde{M}) + K_2^{\mathrm{T}} \Lambda (K_\infty - \widetilde{K})\} \tag{4.4.8}$$

where

$$\begin{aligned}\Xi &= I_p + K_2^{\mathrm{T}} \Lambda K_2 \\ \widetilde{M} &= L Y C^{\mathrm{T}} (R + C Y C^{\mathrm{T}}) \\ \widetilde{K} &= (A Y C^{\mathrm{T}} + B D^{\mathrm{T}})(R + C Y C^{\mathrm{T}})^{-1}\end{aligned}$$

Proof: Since Z belongs to $\mathbf{Z}_{\mathrm{const}}(\gamma)$, there exists a positive semi-definite matrix N satisfying

$$Z V_{11} Z^{\mathrm{T}} + N = \widehat{V}$$

In order to minimize $J_2(Z,Y)$ with respect to Z, Y and N, we form the Lagrangian

$$\begin{aligned}\mathcal{L}(Z,Y,N) = \frac{1}{2}[&J_2(Z,Y) + \mathrm{Tr}\{\Lambda^{\mathrm{T}}\mathrm{Lyap(Z,Y)}\}\\ &+\mathrm{Tr}\{\Psi^{\mathrm{T}}(ZV_{11}Z^{\mathrm{T}} + N - \widehat{V})\}]\end{aligned} \tag{4.4.9}$$

where Λ and Ψ are the costate matrices. As well known, the necessary condition for the optimality is that

$$\frac{\partial\mathcal{L}}{\partial Z} = 0, \quad \frac{\partial\mathcal{L}}{\partial Y} = 0, \quad \frac{\partial\mathcal{L}}{\partial N} = 0$$

By using the formula for the differentiation of the trace of a matrix [1], we obtain (4.4.7) from $\partial\mathcal{L}/\partial Y = 0$. Since $A_\infty + K_2ZC$ is stable, Λ is a unique positive semi-definite solution to the Lyapunov equation (4.4.7). Moreover, we have $\partial\mathcal{L}/\partial N = \Psi = 0$. It then follows that

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial Z} &= \Xi Z(R + CYC^{\mathrm{T}}) + L_\infty YC^{\mathrm{T}} - M_\infty R\\ &+K_2^{\mathrm{T}}\Lambda(A_\infty YC^{\mathrm{T}} + BD^{\mathrm{T}})\end{aligned}$$

Note that Ξ and $R + CYC^{\mathrm{T}}$ are positive definite since Λ, $Y \geq 0$. Hence, we obtain (4.4.8) from the above equation. ■

The simultaneous equations (4.4.6)–(4.4.8) can be solved by the following algorithm based on the gradient method.

Step 0: Set the initial value $\widehat{Z}_0 \in \mathbf{Z}_{\mathrm{const}}(\gamma)$.

Step 1: For $i = 1, 2, \cdots$, find the solutions $\widehat{Y}_i$ and $\widehat{\Lambda}_i$ to the Lyapunov equations:

$$\begin{aligned}\widehat{Y}_i &= (A_\infty + K_2\widehat{Z}_iC)\widehat{Y}_i(A_\infty + K_2\widehat{Z}_iC)^{\mathrm{T}}\\ &+(B_\infty + K_2\widehat{Z}_iD)(B_\infty + K_2\widehat{Z}_iD)^{\mathrm{T}}\end{aligned} \tag{4.4.10}$$

$$\begin{aligned}\widehat{\Lambda}_i &= (A_\infty + K_2\widehat{Z}_iC)^{\mathrm{T}}\widehat{\Lambda}_i(A_\infty + K_2\widehat{Z}_iC)\\ &+(L_\infty + \widehat{Z}_iC)^{\mathrm{T}}(L_\infty + \widehat{Z}_iC)\end{aligned} \tag{4.4.11}$$

Step 2: For a prescribed small constant $\varepsilon > 0$, check the following inequality holds or not.

$$\left[\left\|\frac{\partial\mathcal{L}}{\partial Z}\right\|\right]_{(Z,Y,\Lambda)=(\widehat{Z}_i,\widehat{Y}_i,\widehat{\Lambda}_i)} < \varepsilon$$

If it holds, set $Z := \widehat{Z}_i$ and quit. Otherwise, go to **Step 3.**

Step 3: Update $\widehat{Z}_i$ by

$$\widehat{Z}_{i+1} = \widehat{Z}_i - \delta \left[\frac{\partial \mathcal{L}}{\partial Z}\right]_{(Z,Y,\Lambda)=(\widehat{Z}_i,\widehat{Y}_i,\hat{\Lambda}_i)}$$

with δ small positive constant. Goto **Step 1.**

Remark 4.1: To ensure $\widehat{Z}_0 \in \mathbf{Z}_{\text{const}}(\gamma)$, we can choose $\widehat{Z}_0 = 0$, which implies we can start the algorithm from the central $\mathbf{H}_\infty$ filter. Note also that as γ becomes large, the central $\mathbf{H}_\infty$ filter approaches to the $\mathbf{H}_2$ optimal filter. Thus, the fast convergence to the optimal solution can be achieved by starting from the zero initial value.

Remark 4.2: It may be noted that $A_\infty + K_2ZC$ is stable as long as Z belongs to $\mathbf{Z}_{\text{const}}(\gamma)$. Thus, the solutions to the Lyapunov equations (4.4.10),(4.4.11) exist for any i if we select a sufficiently small δ.

5. Numerical Examples

Example 4.1: We first consider the system given by

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} -0.2 & -0.5 \\ 1.5 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} d_k \\ y_k &= \begin{bmatrix} -2 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 & 1 \end{bmatrix} d_k \\ z_k &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k \end{aligned}$$

By Theorem 4.1, we get $\gamma_X = 0.806$. We also obtain $\gamma_{\text{opt}} = 1.065$. The relationships between γ and the eigenvalues of P and X are illustrated in Figs. 4.1 and 4.2, respectively. We see from the figures that the eigenvalues of P and X are respectively monotonically non-increasing and non-decreasing functions of γ, and that one of the eigenvalues of P diverges to $+\infty$ and reappears from $-\infty$ as γ traverses γ_{opt} from above. Moreover, the eigenvalues of X and P converge to finite values as γ goes to γ_X.

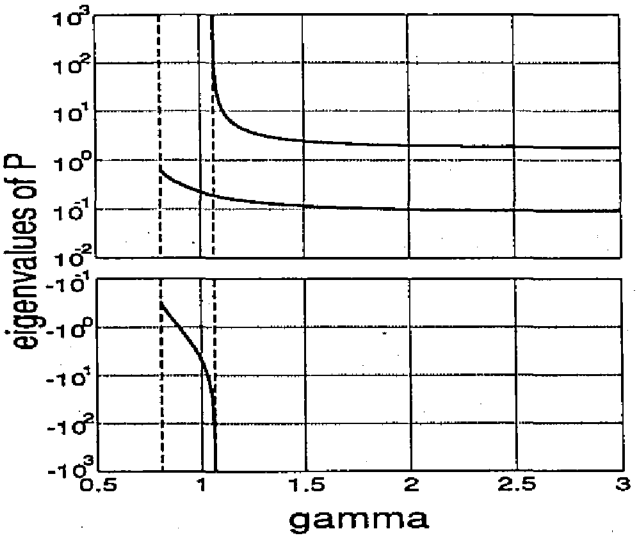


Fig. 4.1: Eigenvalues of P (Example 4.1)

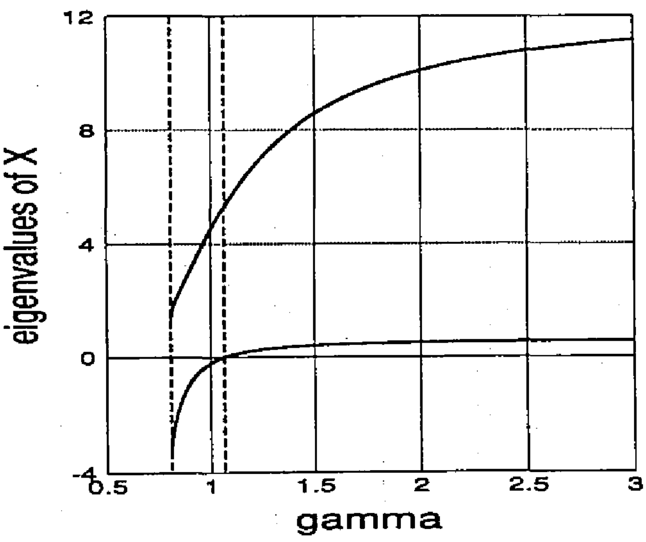


Fig. 4.2: Eigenvalues of X (Example 4.1)

Example 4.2: The second example is the same system that is considered in Chapter 3.

$$x_{k+1} = \begin{bmatrix} 5 & 0.5 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} d_k$$

$$y_k = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} d_k$$

$$z_k = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_k$$

We obtain $\gamma_{\text{opt}} = 3.1120$ and $\gamma_X = 0.4861$ for this example. The upper bound of $\|Z\|_\infty$ given in (4.3.7) is illustrated in Fig. 4.3. As shown in the figure, the upper bound $\sqrt{\lambda_{\max}(\widehat{V})/\lambda_{\min}(V_{11})}$ decreases as γ approaches γ_{opt}. This implies that the size of $\mathbf{Z}(\gamma)$ monotonically decreases as γ decreases. We also see that Theorem 4.7 (i) applies to this example because the degree of freedom does not reduce to zero at γ_{opt}. In fact, it follows from (4.3.7) that $\|Z\|_\infty \leq 0.6261$ when $\gamma = 3.1120$, and taking $Y_{12} = 0.6261$ yields $Z(\sigma) \in \mathbf{Z}_o$ and $\|T_{ed}\|_\infty = 3.1120$.

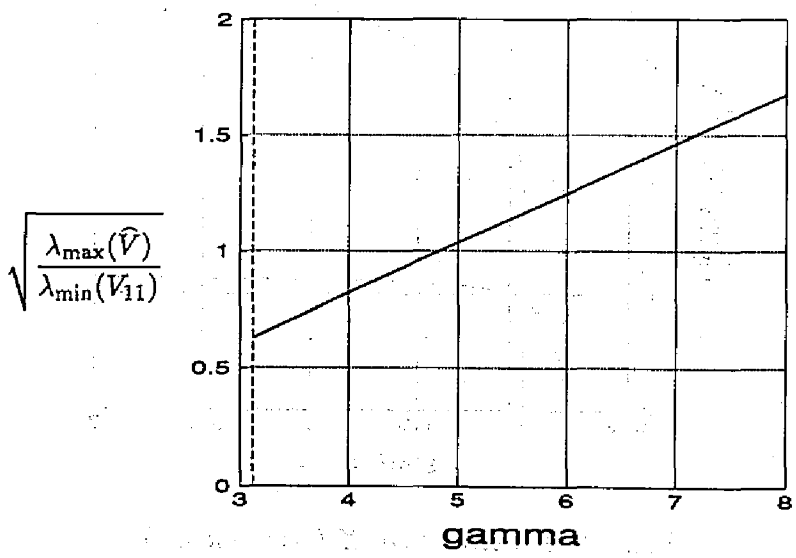


Fig. 4.3: Upper bound of $\|Z\|_\infty$ (Example 4.2)

Example 4.3: As an example of the case (ii) of Theorem 4.7, we consider

$$x_{k+1} = \begin{bmatrix} 5 & 0.5 \\ 0 & 2 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} d_k$$
$$y_k = \begin{bmatrix} 1 & 2 \end{bmatrix} x_k + \begin{bmatrix} 0 & 1 \end{bmatrix} d_k$$
$$z_k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k$$

For this example, we obtain $\gamma_{\text{opt}} = 3.500$ for this example. Fig. 4.4 shows the relationship between γ and $\sqrt{\lambda_{\max}(\widehat{V})/\lambda_{\min}(V_{11})}$. As γ approaches the optimum γ_{opt} from above, the upper bound $\sqrt{\lambda_{\max}(\widehat{V})/\lambda_{\min}(V_{11})}$ converges to zero. Therefore, in this example, the degree of freedom of $T_f(\sigma)$ reduces to zero at γ_{opt}, and hence the optimal $\mathbf{H}_\infty$ filter is uniquely determined as the limit of (2.4.31) with $\gamma \to \gamma_{\text{opt}} + 0$.

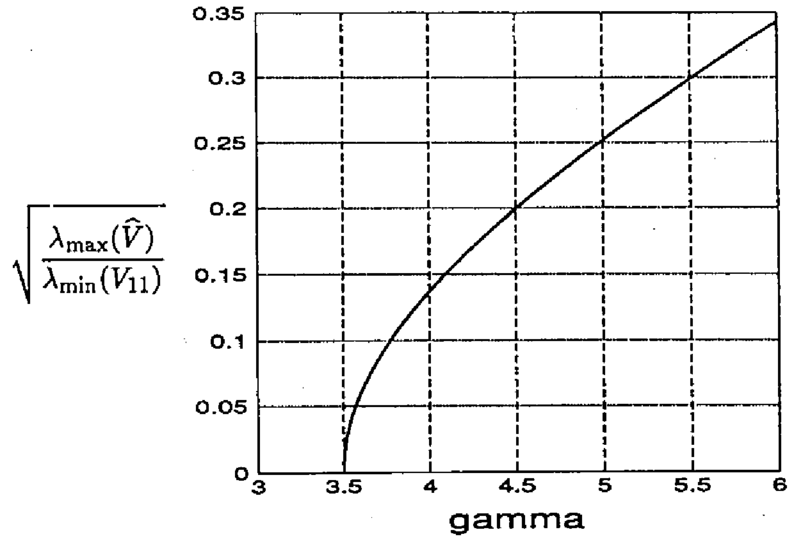


Fig. 4.4: Upper bound of $\|Z\|_\infty$ (Example 4.3)

We next apply the mixed $\mathbf{H}_2/\mathbf{H}_\infty$ filtering algorithm given in Section 4.4 to this example. The relation between γ and the $\mathbf{H}_2$ performance is illustrated in Fig. 4.5. When γ is large, the difference of the $\mathbf{H}_2$ performance between the central filter and the mixed $\mathbf{H}_2/\mathbf{H}_\infty$ filter is very small. This is because as γ goes to infinity, the $\mathbf{H}_\infty$ filtering problem reduces to the $\mathbf{H}_2$ optimal filtering problem and hence the central filter reduces to the

$\mathbf{H}_2$ optimal (Kalman) filter. When γ is near the optimum γ_{opt}, the degree of freedom contained in the $\mathbf{H}_\infty$ filter is very small, and hence the $\mathbf{H}_2$ performances of the two filters are very close as shown in Fig. 4.5. Fig. 4.6 also demonstrates the relationship between γ and $\|Z\|$ of the $\mathbf{H}_2/\mathbf{H}_\infty$ optimal filter. As discussed above, the contribution of the free parameter Z is small when γ is close to γ_{opt}, or when γ is very large.

6. Concluding Remarks

In this chapter, we have examined the behavior of the stabilizing solution of the $\mathbf{H}_\infty$ ARE (4.2.1) with respect to the variation of the prescribed $\mathbf{H}_\infty$ norm bound γ. The following results have been obtained.

The infimum of the parameter γ, for which a stabilizing solution to the $\mathbf{H}_\infty$ ARE exists, is characterized in terms of the $\mathbf{L}_\infty$ norm of a certain transfer matrix. The stabilizing solution $P \in \mathbf{P}(\gamma)$ is a monotonically non-increasing convex function of γ. Moreover, a new parametrization of all $\mathbf{H}_\infty$ filter was derived. Based on the above results, we have shown that the size of the set of all $\mathbf{H}_\infty$ filters is monotonically increasing with respect to $\gamma\,(> \gamma_{\text{opt}})$, and proved that there are possibilities that the degree of freedom of the $\mathbf{H}_\infty$ filter reduces at the optimum γ_{opt}. We also propose an $\mathbf{H}_2/\mathbf{H}_\infty$ filtering algorithm which makes use of the free parameter $Z(\sigma)$. The present results provide a guideline for selecting the values of the parameters γ and $Z(\sigma) \in \mathbf{Z}(\gamma)$. It may be also noted that the analyses in this chapter can be applied to those of the $\mathbf{H}_\infty$ controllers for 2-block problems.

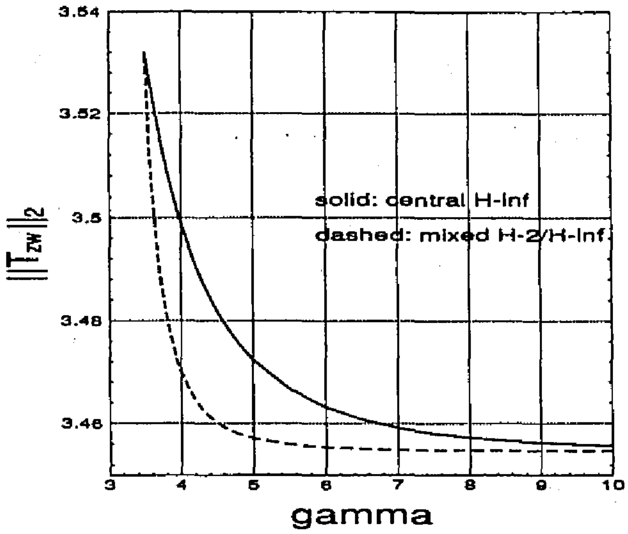


Fig. 4.5: Relation between γ and $\mathbf{H}_2$ performances

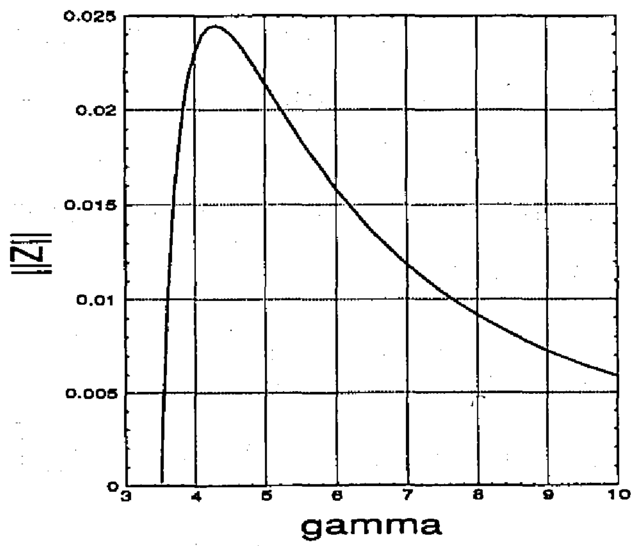


Fig. 4.6: Norm of the optimal Z in the mixed $\mathbf{H}_2/\mathbf{H}_\infty$ sense

Appendix 4.1: Proof of Lemma 4.1

We denote the (F,G)-uncontrollable subspace and the stable (F,G)-uncontrollable subspace by $\mathbf{C}_{F,G}$ and $\mathbf{S}_{F,G}$, respectively. That is,

$$\mathbf{C}_{F,G} = \{x \in \mathbf{C}^n \mid x^{\mathrm{H}}[G\ FG\ \cdots\ F^{n-1}G] = 0\}$$

$$\mathbf{S}_{F,G} = \mathbf{C}_{F,G} \cap \{\bigoplus_{|\lambda|<1} \mathrm{Ker}(\lambda I_n - F)^n\}$$

We first prove $\mathbf{S}_{F,G} \subseteq \mathrm{Ker}P \subseteq \mathbf{C}_{F,G}$. Since P is assumed to be positive semi-definite, we can define $\bar{P} = P(I_n + C^{\mathrm{T}}R^{-1}CP)^{-1} \geq 0$. Then, we get

$$P = F\bar{P}F^{\mathrm{T}} + F\bar{P}L^{\mathrm{T}}\widehat{V}^{-1}L\bar{P}F^{\mathrm{T}} + GG^{\mathrm{T}}$$

Let $\xi \neq 0$ be any element of Ker P. Pre-multiplying the above equation by ξ^{H} and post-multiplying by ξ yield

$$\xi^{\mathrm{H}}F(\bar{P} + \bar{P}L^{\mathrm{T}}\widehat{V}^{-1}L\bar{P})F^{\mathrm{T}}\xi + \xi^{\mathrm{H}}GG^{\mathrm{T}}\xi = 0$$

Since $P \in \mathbf{P}(\gamma)$, $\xi^{\mathrm{H}}FP = 0$ and $\xi^{\mathrm{H}}G = 0$ hold. Thus, by repeating the above argument, we see that $\xi \in \mathbf{C}_{F,G}$. Moreover, let Ξ be a matrix which consists of the bases of $\mathbf{S}_{F,G}$. Then, there exists a stable matrix Λ such that $\Xi^{\mathrm{H}}F = \Lambda\Xi^{\mathrm{H}}$ and $\Xi^{\mathrm{H}}G = 0$ hold. Post-multiplying (4.2.2) by Ξ yields $P\Xi = F_{\mathrm{st}}(P\Xi)\Lambda^{\mathrm{H}}$, where $F_{\mathrm{st}} = F - FP\widehat{C}^{\mathrm{T}}V^{-1}\widehat{C}$. Since F_{st} and Λ^{H} are stable, we get $P\Xi = 0$, i.e., $\mathbf{S}_{F,G} \subseteq \mathrm{Ker}\,P$.

Next, we show $\mathrm{Ker}\,P \cap (\mathbf{C}_{F,G} \ominus \mathbf{S}_{F,G}) = 0$. Since $\mathbf{S}_{F,G} \subsetneq \mathrm{Ker}P \subsetneq \mathbf{C}_{F,G}$, there exists a similarity transformation T such that

$$T^{-1}FT = \begin{bmatrix} F_1 & F_{12} \\ 0 & F_2 \end{bmatrix},\ T^{-1}G = \begin{bmatrix} G_1 \\ 0 \end{bmatrix}$$

$$CT = \begin{bmatrix} C_1 & C_2 \end{bmatrix},\ LT = \begin{bmatrix} L_1 & L_2 \end{bmatrix},\ T^{-1}PT^{-\mathrm{T}} = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where $P_1 > 0$ and (F_1, G_1) has no uncontrollable modes inside the unit disk. By simple calculation, we obtain $F_{\mathrm{st}} = T\begin{bmatrix} * & * \\ 0 & F_2 \end{bmatrix}T^{-1}$ where $*$ denotes irrelevant terms. Since F_{st} is stable, so is F_2. This implies $\mathrm{Ker}\,P \cap (\mathbf{C}_{F,G} \ominus \mathbf{S}_{F,G}) = 0$. Thus, we have proved $\mathrm{Ker}\,P = \mathbf{S}_{F,G}$. ■

Appendix 4.2: Proof of Thorem 4.2

As shown in the proof of Theorem 4.1, there exists a positive definite anti-stabilizing solution X to the ARE (4.2.8) with $\widetilde{V}_0 := I_m - G_1^{\mathrm{T}} X_0 G_1 > 0$.

For $n = 0, 1, 2, \cdots$, we define

$$\widehat{F}_n = F_1 + G_1 \Gamma_n, \quad \Gamma_n = \widetilde{V}_n^{-1} G_1^{\mathrm{T}} X_n F_1$$
$$\widetilde{V}_n = I_m - G_1^{\mathrm{T}} X_n G_1$$

We now show the convergence of the solutions of the Lyapunov equation

$$X_{n+1} = \widehat{F}_n^{\mathrm{T}} X_{n+1} \widehat{F}_n - \Gamma_n^{\mathrm{T}} \Gamma_n - C_1^{\mathrm{T}} R^{-1} C_1 + \gamma^{-2} L_1^{\mathrm{T}} L_1 \tag{A.4.1}$$

The main idea of the proof of convergence is similar to the proof in [38]. If X_n converges to a finite value as n goes to infinity, the limit can be obtained by taking $X_{n+1} = X_n$ and it satisfies (4.2.4). Since $\widehat{F}_0$ is anti-stable, we easily get $X_0 \geq X_1$ and $\widetilde{V}_1 \geq \widetilde{V}_0 > 0$.

We first show that $X_1 \geq X_{\mathrm{a}}$ holds for all $X_{\mathrm{a}} \in \mathbf{X}(\gamma)$. Simple algebra yields

$$\begin{aligned} X_{\mathrm{a}} = {} & \widehat{F}_{n-1}^{\mathrm{T}} X_{\mathrm{a}} \widehat{F}_{n-1} - \Gamma_{n-1}^{\mathrm{T}} \Gamma_{n-1} - C_1^{\mathrm{T}} R^{-1} C_1 + \gamma^{-2} L_1^{\mathrm{T}} L_1 \\ & + (\Gamma_{\mathrm{a}} - \Gamma_{n-1})^{\mathrm{T}} \widetilde{V}_{\mathrm{a}} (\Gamma_{\mathrm{a}} - \Gamma_{n-1}) \end{aligned} \tag{A.4.2}$$

where $\widetilde{V}_{\mathrm{a}} = I_m - G_1^{\mathrm{T}} X_{\mathrm{a}} G_1$ and $\Gamma_{\mathrm{a}} = \widetilde{V}_{\mathrm{a}}^{-1} G_1^{\mathrm{T}} X_{\mathrm{a}} F_1$. Thus, from (A.4.1) and (A.4.2), we get

$$X_1 - X_{\mathrm{a}} = \widehat{F}_0^{\mathrm{T}} (X_1 - X_{\mathrm{a}}) \widehat{F}_0 - (\Gamma_{\mathrm{a}} - \Gamma_0)^{\mathrm{T}} \widetilde{V}_{\mathrm{a}} (\Gamma_{\mathrm{a}} - \Gamma_0)$$

Since $\widehat{F}_0$ is anti-stable and $\widetilde{V}_{\mathrm{a}} > 0$, by Lyapunov's theorem, $X_1 - X_{\mathrm{a}} \geq 0$ holds.

Suppose that $\widehat{F}_{n-1}$ is anti-stable for $n = 1, 2, \cdots, k$, and that

$$X_0 \geq X_1 \geq \cdots \geq X_k \geq X_{\mathrm{a}}, \quad \forall X_{\mathrm{a}} \in \mathbf{X}(\gamma)$$

It is straightforward to show that

$$\begin{aligned} X_k = {} & \widehat{F}_k^{\mathrm{T}} X_k \widehat{F}_k - \Gamma_k^{\mathrm{T}} \Gamma_k - C_1^{\mathrm{T}} R^{-1} C_1 + \gamma^{-2} L_1^{\mathrm{T}} L_1 \\ & - (\Gamma_k - \Gamma_{k-1})^{\mathrm{T}} \widetilde{V}_k (\Gamma_k - \Gamma_{k-1}) \end{aligned} \tag{A.4.3}$$

Thus, we have

$$
\begin{aligned}
X_k - X_\mathrm{a} = \widehat{F}_k^\mathrm{T}(X_k - X_\mathrm{a})\widehat{F}_k - (\Gamma_\mathrm{a} - \Gamma_k)^\mathrm{T}\widetilde{V}_\mathrm{a}(\Gamma_\mathrm{a} - \Gamma_k) \\
-(\Gamma_k - \Gamma_{k-1})^\mathrm{T}\widetilde{V}_k(\Gamma_k - \Gamma_{k-1})
\end{aligned}
\tag{A.4.4}
$$

We here assume that $\widehat{F}_k$ has an eigenvalue λ with $|\lambda| \le 1$. Then, there exists a nonzero vector η such that $\widehat{F}_k\eta = \lambda\eta$. It thus follows from (A.4.4) that

$$
\begin{aligned}
(1 - |\lambda|^2)\eta^\mathrm{H}(X_k - X_\mathrm{a})\eta = -\eta^\mathrm{H}(\Gamma_\mathrm{a} - \Gamma_k)^\mathrm{T}\widetilde{V}_\mathrm{a}(\Gamma_\mathrm{a} - \Gamma_k)\eta \\
-\eta^\mathrm{H}(\Gamma_k - \Gamma_{k-1})^\mathrm{T}\widetilde{V}_k(\Gamma_k - \Gamma_{k-1})\eta
\end{aligned}
$$

Since $X_k \ge X_\mathrm{a}$ and $\widetilde{V}_\mathrm{a} \ge \widetilde{V}_k > 0$, the both sides of the above equation must be zero. Thus we get $(\Gamma_k - \Gamma_{k-1})\eta = 0$. In this case, we see $\widehat{F}_{k-1}\eta = \widehat{F}_k\eta = \lambda\eta$. This contradicts the fact that $\widehat{F}_{k-1}$ is anti-stable. Consequently, $\widehat{F}_k$ is also anti-stable.

Furthermore, since

$$
\begin{aligned}
X_k - X_{k+1} &= \widehat{F}_k^\mathrm{T}(X_k - X_{k+1})\widehat{F}_k - \Gamma_k^\mathrm{T}\Gamma_k - (\Gamma_k - \Gamma_{k-1})^\mathrm{T}\widetilde{V}_k(\Gamma_k - \Gamma_{k-1}) \\
X_{k+1} - X_\mathrm{a} &= \widehat{F}_k^\mathrm{T}(X_{k+1} - X_\mathrm{a})\widehat{F}_k - (\Gamma_\mathrm{a} - \Gamma_k)^\mathrm{T}\widetilde{V}_\mathrm{a}(\Gamma_\mathrm{a} - \Gamma_k)
\end{aligned}
$$

holds from (A.4.1),(A.4.2) and (A.4.3), we get $X_k \ge X_{k+1} \ge X_\mathrm{a}$. It follows by induction that X_n is monotonically non-increasing and bounded below. Therefore, X_n converges to a maximal element X of $\mathbf{X}(\gamma)$, and $\widehat{F} := F_1 + G_1\widetilde{V}^{-1}G_1^\mathrm{T}XF_1$ has no eigenvalues inside the open unit disk.

Chapter 5

A Game Theoretic Approach to $\mathbf{H}_\infty$ Filtering Problem

1. Introduction

Chapter 2 has given a solution to the infinite-horizon $\mathbf{H}_\infty$ filtering problem for time-invariant systems based on the model matching technique in the frequency domain. Since $\mathbf{H}_\infty$ norm is the $\mathbf{L}_2$ induced norm of a system, $\mathbf{H}_\infty$ optimization problem is a kind of minimax optimization problems. In other words, the $\mathbf{H}_\infty$ filtering problem is the minimax optimization problem of minimizing the maximum of the energy in the estimation errors over all possible disturbance trajectories. However, the model matching approach given in Chapter 2 does not directly provide such minimax properties of the $\mathbf{H}_\infty$ filtering problem since it merely minimize the largest singular value of a certain transfer matrix. In order to make clear the minimax aspect of the $\mathbf{H}_\infty$ filtering problem, it is essential to adopt a difference game approach in the time-domain.

In this chapter, we will consider the finite-horizon minimax state estimation problems which are closely related with the $\mathbf{H}_\infty$ filtering and prediction problems. We first derive necessary conditions for the existence of the minimax solutions by exploiting the sweep method, which is a straightforward optimization method based on the Lagrange multiplier technique [4]. Sufficient conditions for the existence of the minimax solutions are given based on the square completion technique. It is shown that the optimal minimax state estimators are identical to the central $\mathbf{H}_\infty$ filter and $\mathbf{H}_\infty$ predictor.

2. Problem Formulation

In this chapter, we consider a linear time-varying system described by

$$x_{k+1} = A_k x_k + B_k w_k \tag{5.2.1}$$

$$y_k = C_k x_k + D_k v_k \tag{5.2.2}$$

where $x_k \in \mathbf{R}^n$, $y_k \in \mathbf{R}^q$ are the state vector and the measurement. The exogenous inputs $w_k \in \mathbf{R}^m$ and $v_k \in \mathbf{R}^q$ are the process disturbance and the measurement noise, respectively. Note that the values of w_k, v_k are unknown while w_k, v_k are arbitrary $\mathbf{L}_2[0,N]$ signals. Moreover, we assume that D_k is nonsingular, so that $R_k := D_k D_k^{\mathrm{T}} > 0$ holds.

It may be noted that the system (5.2.1),(5.2.2) is different from the system (2.2.1),(2.2.2) considered in the previous chapters. However, in the time-invariant case, the filtering problem for (2.2.1),(2.2.2) can be reduced to the problem for (5.2.1),(5.2.2) under the assumption that D in (2.2.2) is right invertible. For the detail, see Appendix 5.1.

As well as estimating x_k, we wish to estimate the vector $z_k \in \mathbf{R}^p$ defined by

$$z_k = L_k x_k \tag{5.2.3}$$

Let $\hat{z}_k$ be the estimate of z_k based on $\{y_0, \cdots, y_k\}$. Moreover, without loss of generality, we assume that the estimate of the initial state x_0 is a priori given by $\bar{x}_0$.

In this chapter, we will discuss the minimax filtering and prediction problems which are closely related with the $\mathbf{H}_\infty$ filtering and prediction problems.

We first define the cost function for the minimax filtering problem. The estimate $\hat{z}_k$ tries to minimize the squared estimation error $\sum_{k=0}^{N} \|z_k - \hat{z}_k\|^2$, while the triple $(\bar{x}_0, w_k, v_k)$ tries to maximize the squared estimation error. Since arbitrary large values of $\|w_k\|$, $\|v_k\|$ and $\|x_0\|$ cause arbitrary large value of the estimation error, we define the cost function J as follows:

$$J(\hat{z}; x_0, w, v) = \sum_{k=0}^{N} \|z_k - \hat{z}_k\|^2 - \gamma^2 \Big(\sum_{k=0}^{N} \|w_k\|^2 + \sum_{k=0}^{N} \|v_k\|^2 + \|x_0 - \bar{x}_0\|^2_{\Pi^{-1}}\Big) \tag{5.2.4}$$

The second term in the right-hand side is the penalty term on w_k, v_k and x_0; γ is a positive constant which represents the magnitude of the penalty. The weighting matrix Π is positive definite and represents the uncertainty of the initial state x_0. From the game theoretic viewpoint, we can say that the filtered estimate $\widehat{z}_k$ and the triple (w_k, v_k, x_0) are the minimizing and maximizing polices of J, respectively.

The finite-horizon $\mathbf{H}_\infty$ filtering problem is to find estimates $\widehat{x}_k$ and $\widehat{z}_k$ satisfying

$$\sup_{w,v,x_0} \frac{\sum_{k=0}^{N} \|z_k - \widehat{z}_k\|^2}{\sum_{k=0}^{N}(\|w_k\|^2 + \|v_k\|^2) + \|x_0 - \bar{x}_0\|^2_{\Pi^{-1}}} < \gamma^2 \tag{5.2.5}$$

This condition is equivalent to

$$\begin{aligned} J(\widehat{z}; x_0, w, v) < 0, \quad & \forall (x_0, w_k, v_k) \text{ s.t.} \\ & \sum_{k=0}^{N}(\|w_k\|^2 + \|v_k\|^2) + \|x_0 - \bar{x}_0\|^2_{\Pi^{-1}} \neq 0 \end{aligned} \tag{5.2.6}$$

Therefore, the minimax estimation problems formulated here are closely related to the finite-horizon $\mathbf{H}_\infty$ filtering problem.

By (5.2.2), we easily see that

$$v_k = D_k^{-1}(y_k - C_k x_k)$$

Thus, we rewrite the cost function J as

$$\begin{aligned} J(\widehat{z}; x_0, w, v) = & \sum_{k=0}^{N} \|z_k - \widehat{z}_k\|^2 - \gamma^2 (\sum_{k=0}^{N} \|w_k\|^2 \\ & + \sum_{k=0}^{N} \|y_k - C_k x_k\|^2_{R_k^{-1}} + \|x_0 - \bar{x}_0\|^2_{\Pi^{-1}}) \end{aligned} \tag{5.2.7}$$

Thus the minimax problem between $\widehat{z}_k$ and (x_0, w_k, v_k) reduces to the problem between $\widehat{z}_k$ and (x_0, w_k, y_k).

We denote the optimal policies by $\widehat{z}_k^*$ and (x_0^*, w_k^*, y_k^*), respectively. We call w_k^* the worst-case disturbance. Also let v_k^* be the worst-case noise corresponding to y_k^*. The quadruple $(\widehat{z}_k^*, x_0^*, w_k^*, v_k^*)$ are referred to as the optimal solution of the minimax problem.

In this chapter, we consider two kinds of minimax problems. In the first problem, the measurement set $\{y_0, \cdots, y_{k-1}, y_k\}$ is available for the estimation at time k. We call this problem "a filtering problem". The second problem is called "a one-step prediction problem" or merely "a prediction problem" since $\{y_0, \cdots, y_{k-1}\}$ rather than $\{y_0, \cdots, y_{k-1}, y_k\}$

is available at time k. It may be noted that the problem in which all the measurement $\{y_0, \cdots, y_N\}$ are available for the estimation at any time $k \in [0, N]$ is called "a fixed-interval smoothing problem". A remark on this minimax smoothing problem is given in Appendix 5.2.

For the problem of filtering case, since y_k is available for $\widehat{z}_k$, the order of the minimax optimization is

$$\max_{y_N}(\min_{\widehat{z}_N}(\max_{w_N} \cdots \max_{y_k}(\min_{\widehat{z}_k}(\max_{w_k} \\ \cdots \max_{y_0}(\min_{\widehat{z}_k}(\max_{w_0,x_0} J)) \cdots)) \cdots)) \tag{5.2.8}$$

Similarly, since y_k is not available for $\widehat{z}_k$, the minimax prediction problem is formulated by

$$\min_{\widehat{z}_N}(\max_{y_N}(\max_{w_N} \cdots \min_{\widehat{z}_k}(\max_{y_k}(\max_{w_k} \\ \cdots \min_{\widehat{z}_k}(\max_{y_0}(\max_{w_0,x_0} J)) \cdots)) \cdots)) \tag{5.2.9}$$

Remark 5.1: In [40] and [54], the $\mathbf{H}_\infty$ filter and $\mathbf{H}_\infty$ predictor were derived from the saddle-point policies for the minimax state estimation problems with different cost functions:

$$J = \sum_{k=1}^{N} \|z_k - \widehat{z}_k\|^2 - \gamma^2\{\sum_{k=1}^{N}(\|w_{k-1}\|^2 + \|v_k\|^2) + \|x_0 - \bar{x}_0\|^2_{\Pi^{-1}}\} \quad \text{(filtering problem)}$$

$$J = \sum_{k=1}^{N} \|z_k - \widehat{z}_k\|^2 - \gamma^2\{\sum_{k=1}^{N}(\|w_{k-1}\|^2 + \|v_{k-1}\|^2) + \|x_0 - \bar{x}_0\|^2_{\Pi^{-1}}\} \quad \text{(prediction problem)}$$

Unlike the above approach, we will show that both central $\mathbf{H}_\infty$ filter and $\mathbf{H}_\infty$ predictor can be derived from the same cost function (5.2.4).

3. Necessary Conditions

3.1 Maximizing with respect to x_0 and w_k

Since w_k is an arbitrary $\mathbf{L}_2[0, N]$ signal, without loss of generality, it can be assumed that w_k can utilize all the data of $\{y_0, \cdots, y_N\}$ and $\{\widehat{z}_0, \cdots, \widehat{z}_N\}$. Therefore, we can first perform the optimization with respect to x_0 and $\{w_0, \cdots, w_N\}$.

To maximize J with respect to x_0 and w_k, we form the Hamiltonian

$$H_k = \frac{1}{2}\gamma^{-2}\{\|L_k x_k - \widehat{z}_k\|^2 - \gamma^2(\|w_k\|^2 + \|y_k - C_k x_k\|^2_{R_k^{-1}})\}$$
$$+\lambda_{k+1}^{\mathrm{T}}(A_k x_k + B_k w_k - x_{k+1}) \tag{5.3.1}$$

where λ_k is the costate vector. The cost function J is related to H_k by

$$\frac{1}{2}\gamma^{-2}J = \sum_{k=0}^{N} H_k - \frac{1}{2}\|x_0 - \bar{x}_0\|^2_{\Pi^{-1}}$$
$$- \sum_{k=0}^{N} \lambda_{k+1}^{\mathrm{T}}(A_k x_k + B_k w_k - x_{k+1}) \tag{5.3.2}$$

Let (x_k^*, λ_k^*) be the trajectories of (x_k, λ_k) corresponding to the worst case disturbance w_k^*. Then, the necessary conditions of optimality is given by

$$0 = \left.\frac{\partial H_k}{\partial \lambda_{k+1}}\right|_{(w_k, x_k, \lambda_{k+1}) = (w_k^*, x_k^*, \lambda_{k+1}^*)} \tag{5.3.3}$$

$$0 = \left.\frac{\partial H_k}{\partial w_k}\right|_{(w_k, x_k, \lambda_{k+1}) = (w_k^*, x_k^*, \lambda_{k+1}^*)} \tag{5.3.4}$$

$$\lambda_k^* = \left.\frac{\partial H_k}{\partial x_k}\right|_{(w_k, x_k, \lambda_{k+1}) = (w_k^*, x_k^*, \lambda_{k+1}^*)}, \quad \lambda_{N+1}^* = 0 \tag{5.3.5}$$

$$\Pi^{-1}(x_0^* - \bar{x}_0) = \left.\frac{\partial H_0}{\partial x_0}\right|_{(w_0, x_0, \lambda_1) = (w_0^*, x_0^*, \lambda_1^*)} = \lambda_0^* \tag{5.3.6}$$

Therefore, we have

$$x_{k+1}^* = A_k x_k^* + B_k w_k^*, \quad x_0^* = \bar{x}_0 + \Pi\lambda_0^* \tag{5.3.7}$$

$$w_k^* = B_k^{\mathrm{T}}\lambda_{k+1}^* \tag{5.3.8}$$

$$A_k^{\mathrm{T}}\lambda_{k+1}^{\mathrm{T}} = \lambda_k^* + (C_k^{\mathrm{T}}R_k^{-1}C_k - \gamma^{-2}L_k^{\mathrm{T}}L_k)x_k^* - C_k^{\mathrm{T}}R_k^{-1}y_k + L_k^{\mathrm{T}}\widehat{z}_k \tag{5.3.9}$$

$$= \lambda_k^* - C_k^{\mathrm{T}}R_k^{-1}(y_k - C_k x_k^*) + \gamma^{-2}L_k^{\mathrm{T}}(\widehat{z}_k - L_k x_k^*), \quad \lambda_{N+1}^* = 0 \tag{5.3.10}$$

From (5.3.7)–(5.3.9), we have the two point boundary values problem (TPBVP)

$$\begin{bmatrix} I_n & -B_k B_k^{\mathrm{T}} \\ 0 & A_k^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x_{k+1}^* \\ \lambda_{k+1}^* \end{bmatrix} = \begin{bmatrix} A_k & 0 \\ C_k^{\mathrm{T}}R_k^{-1}C_k - \gamma^{-2}L_k^{\mathrm{T}}L_k & I_n \end{bmatrix} \begin{bmatrix} x_k^* \\ \lambda_k^* \end{bmatrix}$$
$$+ \begin{bmatrix} 0 \\ -C_k^{\mathrm{T}}R_k^{-1}y_k + \gamma^{-2}L_k^{\mathrm{T}}\widehat{z}_k \end{bmatrix}, \quad \begin{cases} x_0^* = \bar{x}_0 + \Pi\lambda_0^* \\ \lambda_{N+1}^* = 0 \end{cases} \tag{5.3.11}$$

Since this TPBVP is non-homogeneous and linear with respect to x_k^* and λ_k^*, x_k^* can be expressed as

$$x_k^* = \widehat{x}_k + P_k \lambda_k^* \tag{5.3.12}$$

Then, from (5.3.11) and (5.3.12), we get

$$\widehat{x}_{k+1} - A_k \widehat{x}_k = (B_k B_k^{\mathrm{T}} - P_{k+1})\lambda_{k+1}^* + A_k P_k \lambda_k^* \tag{5.3.13}$$

$$A_k^{\mathrm{T}} \lambda_{k+1}^* = \Sigma_k \lambda_k^* - C_k^{\mathrm{T}} R_k^{-1}(y_k - C_k \widehat{x}_k) + \gamma^{-2} L_k^{\mathrm{T}}(\widehat{z}_k - L_k \widehat{x}_k) \tag{5.3.14}$$

where

$$\Sigma_k = I_n + (C_k^{\mathrm{T}} R_k^{-1} C_k - \gamma^{-2} L_k^{\mathrm{T}} L_k) P_k$$

Since λ_k^* is finite, Σ_k is nonsingular. It thus follows from (5.3.13) and (5.3.14) that

$$\begin{aligned} \widehat{x}_{k+1} - A_k \widehat{x}_k & - A_k P_k \Sigma_k^{-1} C_k^{\mathrm{T}} R_k^{-1}(y_k - C_k \widehat{x}_k) \\ & + \gamma^{-2} A_k P_k \Sigma_k^{-1} L_k^{\mathrm{T}}(\widehat{z}_k - L_k \widehat{x}_k) \\ = & \ (A_k P_k \Sigma_k^{-1} A_k^{\mathrm{T}} + B_k B_k^{\mathrm{T}} - P_{k+1})\lambda_{k+1}^* \end{aligned}$$

Since the above equation is true for arbitrary λ_k^*, we obtain

$$P_{k+1} = A_k P_k \Sigma_k^{-1} A_k^{\mathrm{T}} + B_k B_k^{\mathrm{T}}, \quad P_0 = \Pi \tag{5.3.15}$$

$$\begin{aligned} \widehat{x}_{k+1} = & \ A_k \widehat{x}_k + A_k P_k \Sigma_k^{-1} C_k^{\mathrm{T}} R_k^{-1}(y_k - C_k \widehat{x}_k) \\ & - \gamma^{-2} A_k P_k \Sigma_k^{-1} L_k^{\mathrm{T}}(\widehat{z}_k - L_k \widehat{x}_k), \quad \widehat{x}_0 = \bar{x}_0 \end{aligned} \tag{5.3.16}$$

The equation (5.3.15) is the well-known $\mathbf{H}_\infty$-type RDE.

3.2 Minimax optimization with respect to $\widehat{z}_k$ and y_k

Simple but tedious calculation using (5.3.7)–(5.3.16) yields

$$\begin{aligned} & \gamma^2(\lambda_{k+1}^{*\mathrm{T}} P_{k+1} \lambda_{k+1}^* - \lambda_k^{*\mathrm{T}} P_k \lambda_k^*) \\ & \quad = -\|\widehat{z}_k - L_k x_k^*\|^2 + \gamma^2(\|w_k^*\|^2 + \|y_k - C_k x_k^*\|^2_{R_k^{-1}}) \\ & \qquad + \begin{bmatrix} \bar{z}_k \\ \bar{y}_k \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Omega_k & -L_k \Xi_k C_k^{\mathrm{T}} R_k^{-1} \\ -R_k^{-1} C_k \Xi_k L_k^{\mathrm{T}} & -R_k^{-1} \bar{\Omega}_k R_k^{-1} \end{bmatrix} \begin{bmatrix} \bar{z}_k \\ \bar{y}_k \end{bmatrix} \end{aligned}$$

where

$$\Omega_k = I_p + \gamma^{-2} L_k \Xi_k L_k^{\mathrm{T}}$$

$$\bar{\Omega}_k = \gamma^2 (R_k - C_k \Xi_k C_k^{\mathrm{T}})$$

$$\bar{y}_k = y_k - C_k \widehat{x}_k$$

$$\bar{z}_k = \widehat{z}_k - L_k \widehat{x}_k$$

$$\Xi_k = P_k \Sigma_k^{-1}$$

Since $\lambda_{N+1}^* = 0$, $x_0^* = \bar{x}_0 + \Pi \lambda_0^*$ and $P_0 = \Pi$, we get

$$\begin{aligned}\sum_{k=0}^{N} \gamma^2 (\lambda_{k+1}^{*\mathrm{T}} P_{k+1} \lambda_{k+1}^* - \lambda_k^{*\mathrm{T}} P_k \lambda_k^*) &= \gamma^2 (\lambda_{N+1}^{*\mathrm{T}} P_{k+1} \lambda_{N+1}^* - \lambda_0^{*\mathrm{T}} P_0 \lambda_0^*) \\ &= -\gamma^2 \|x_0^* - \bar{x}_0\|_{\Pi^{-1}}^2\end{aligned}$$

It thus follows that

$$\begin{aligned}\max_{w, x_0} J(\widehat{z}; x_0, w, v) &= J(\widehat{z}; x_0^*, w^*, v) \\ &= \sum_{k=0}^{N} \{\|L_k x_k^* - \widehat{z}_k\|^2 - \gamma^2 (\|w_k^*\|^2 + \|y_k - C_k x_k^*\|_{R_k^{-1}}^2)\} \\ &\quad -\gamma^2 \|x_0^* - \bar{x}_0\|_{\Pi^{-1}}^2 \\ &= \sum_{k=0}^{N} \begin{bmatrix} \bar{z}_k \\ \bar{y}_k \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Omega_k & -L_k \Xi_k C_k^{\mathrm{T}} R_k^{-1} \\ -R_k^{-1} C_k \Xi_k L_k^{\mathrm{T}} & -R_k^{-1} \bar{\Omega}_k R_k^{-1} \end{bmatrix} \begin{bmatrix} \bar{z}_k \\ \bar{y}_k \end{bmatrix} \qquad (5.3.17)\end{aligned}$$

Minimax Filtering Problem

It is easily seen from (5.3.17) that there exists a unique optimal minimizing policy $\widehat{z}_k^*$ if and only if

$$\Omega_k = I_p + \gamma^{-2} L_k \Xi_k L_k^{\mathrm{T}} > 0 \quad \forall k \in [0, N]$$

Lemma 5.1: *Suppose that $\Omega_k > 0$ and $P_k \geq 0$ hold for the RDE (5.3.15). Then $P_{k+1} \geq 0$ holds.*

Proof: Since P_k is positive semi-definite, there exists a matrix $\bar{P}_k := P_k (I_n + C_k^{\mathrm{T}} R_k^{-1} C_k P_k)^{-1}$. Then, using the matrix inversion lemma, Ω_k can be expressed as

$$\Omega_k = (I_p - \gamma^{-2} L_k \bar{P}_k L_k^{\mathrm{T}})^{-1}$$

Moreover, We define $K_k = P_k C_k^{\mathrm{T}} (R_k + C_k P_k C_k^{\mathrm{T}})^{-1}$ to get

$$\bar{P}_k = (I_n - K_k C_k) P_k (I_n - K_k C_k)^{\mathrm{T}} + K_k R_k K_k^{\mathrm{T}} \geq 0$$

Hence, from the assumption $\Omega_k > 0$, we obtain

$$\begin{aligned} P_{k+1} &= A_k P_k \Sigma_k^{-1} A_k^{\mathrm{T}} + B_k B_k^{\mathrm{T}} \\ &= A_k (\bar{P}_k + \gamma^{-2} \bar{P}_k L_k^{\mathrm{T}} \Omega_k L_k \bar{P}_k) A_k^{\mathrm{T}} + B_k B_k^{\mathrm{T}} \geq 0 \end{aligned}$$

This completes the proof. ■

In the following, we assume that $\Omega_k > 0$ holds for all $k \in [0, N]$. Completing the square with respect to $\bar{z}_k$, (5.3.17) reduces to

$$\begin{aligned} J(\widehat{z}; x_0^*, w^*, y) = &\sum_{k=0}^{N} (\bar{z}_k - \Omega_k^{-1} L_k \Xi_k C_k^{\mathrm{T}} R_k^{-1} \bar{y}_k)^{\mathrm{T}} \Omega_k (\bar{z}_k - \Omega_k^{-1} L_k \Xi_k C_k^{\mathrm{T}} R_k^{-1} \bar{y}_k) \\ &- \gamma^2 \sum_{k=0}^{N} \bar{y}_k^{\mathrm{T}} (R_k + C_k P_k C_k^{\mathrm{T}})^{-1} \bar{y}_k \end{aligned} \tag{5.3.18}$$

Therefore, taking

$$\bar{z}_k - \Omega_k^{-1} L_k \Xi_k C_k^{\mathrm{T}} R_k^{-1} \bar{y}_k = 0 \tag{5.3.19}$$

yields the optimal estimate $\widehat{z}_k^*$. Let $\widehat{x}_{k/t}$ be an estimate of x_k based on the measurement set $\{y_0, \cdots, y_t\}$. Since $\widehat{x}_k$ can be regarded as an estimate of x_k based on $\{y_0, \cdots, y_{k-1}\}$ from (5.3.16), we rewrite as $\widehat{x}_{k/k-1} = \widehat{x}_k$. It then follows that

$$\widehat{z}_k^* = L_k \widehat{x}_{k/k-1} - L_k K_k (y_k - C_k \widehat{x}_{k/k-1}) = L_k \widehat{x}_{k/k} \tag{5.3.20}$$

$$\widehat{x}_{k/k} = \widehat{x}_{k/k-1} + K_k (y_k - C_k \widehat{x}_{k/k-1}) \tag{5.3.21}$$

$$\begin{aligned} \widehat{x}_{k+1/k} &= A_k \widehat{x}_{k/k-1} + A_k K_k (y_k - C_k \widehat{x}_{k/k-1}) \\ &= A_k \widehat{x}_{k/k}, \quad \widehat{x}_{0/-1} = \bar{x}_0 \end{aligned} \tag{5.3.22}$$

$$K_k = P_k C_k^{\mathrm{T}} (R_k + C_k P_k C_k^{\mathrm{T}})^{-1} \tag{5.3.23}$$

If $\Omega_k > 0$ holds, then $P_k \geq 0$ and $R_k + C_k P_k C_k^{\mathrm{T}} > 0$ hold from Lemma 5.1 and $P_0 = \Pi > 0$. Then, by taking $\bar{y}_k = 0$, we get a unique worst case measurement y_k^*. Thus the worst case noise is given by

$$v_k^* = -D_k^{-1} C_k (x_k - \widehat{x}_{k/k-1})$$

Moreover, the cost for the optimal solution is given by

$$J(\widehat{z}^{*}; x_0^{*}, w^{*}, v^{*}) = 0$$

Theorem 5.1: *Consider the minimax filtering problem (5.2.8). For the existence of a unique solution to the problem, it is necessary that the RDE (5.3.15) has a positive semi-definite solution P_k and $\gamma^2 I_p - L_k \bar{P}_k L_k^{\mathrm{T}} > 0$ holds for all $k \in [0, N]$. Then the optimal estimate $\widehat{z}_k^{*}$ is given by (5.3.20)– (5.3.23).*

Minimax One-Step Prediction Problem

We see from (5.3.17) that there exists a unique worst case measurement y_k^{*} if and only if

$$\bar{\Omega}_k = \gamma^2 (R_k - C_k P_k \Sigma_k^{-1} C_k^{\mathrm{T}}) > 0 \quad \forall k \in [0, N]$$

By completing the square with respect to $\bar{y}_k$, (5.3.17) reduces to

$$\begin{aligned} J(\widehat{z}; x_0^{*}, w^{*}, y) = &- \sum_{k=0}^{N} (R_k^{-1}\bar{y}_k + \bar{\Omega}_k^{-1} C_k \Xi_k L_k^{\mathrm{T}} \bar{z}_k)^{\mathrm{T}} \bar{\Omega}_k (R_k^{-1}\bar{y}_k + \bar{\Omega}_k^{-1} C_k \Xi_k L_k^{\mathrm{T}} \bar{z}_k) \\ &+ \sum_{k=0}^{N} \bar{z}_k^{\mathrm{T}} (I_p - \gamma^{-2} L_k P_k L_k^{\mathrm{T}})^{-1} \bar{z}_k \end{aligned} \tag{5.3.24}$$

Thus we get the worst case measurement y_k^{*} by taking

$$R_k^{-1}\bar{y}_k + \bar{\Omega}_k^{-1} C_k \Xi_k L_k^{\mathrm{T}} \bar{z}_k = 0 \tag{5.3.25}$$

Moreover we get

$$J(\widehat{z}; x_0^{*}, w^{*}, y^{*}) = \sum_{k=0}^{N} \bar{z}_k^{\mathrm{T}} (I_p - \gamma^{-2} L_k P_k L_k^{\mathrm{T}})^{-1} \bar{z}_k \tag{5.3.26}$$

Therefore, for the existence of the optimal estimate $\widehat{z}_k^{*}$, it is necessary that

$$\gamma^2 I_p - L_k P_k L_k^{\mathrm{T}} > 0 \tag{5.3.27}$$

Then we get the optimal estimate $\widehat{z}_k^{*}$ by taking $\bar{z}_k = 0$.

$$\widehat{z}_k^{*} = L_k \widehat{x}_{k/k-1} \tag{5.3.28}$$

$$\widehat{x}_{k+1/k} = A_k \widehat{x}_{k/k-1} + A_k \tilde{K}_k (y_k - C_k \widehat{x}_{k/k-1}), \quad \widehat{x}_{0/-1} = \bar{x}_0 \tag{5.3.29}$$

$$\tilde{K}_k = \widetilde{P}_k C_k^{\mathrm{T}} (R_k + C_k \widetilde{P}_k C_k^{\mathrm{T}})^{-1} \tag{5.3.30}$$

Moreover, from (5.3.25), the worst case noise v_k^* is given by

$$v_k^* = -D_k^{-1}C_k(x_k - \widehat{x}_{k/k-1})$$

Then the cost function becomes

$$J(\widehat{z}^*; x_0^*, w^*, v^*) = 0$$

The next lemma shows that $\bar{\Omega}_k > 0$ holds if $\gamma^2 I_p - L_k P_k L_k^{\mathrm{T}} > 0$.

Lemma 5.2: *For the RDE (5.3.15), if $P_k \geq 0$ and $\gamma^2 I_p - L_k P_k L_k^{\mathrm{T}} > 0$ hold, then $\bar{\Omega}_k > 0$ and $P_{k+1} \geq 0$.*

Proof: Since $P_k \geq 0$ and $\gamma^2 I_p - L_k P_k L_k^{\mathrm{T}} > 0$, there exists a symmetric matrix $\widetilde{P}_k$ such that

$$\begin{aligned}\widetilde{P}_k &= P_k(I_n - \gamma^{-2}L_k^{\mathrm{T}}L_kP_k)^{-1} \\ &= P_k + P_kL_k^{\mathrm{T}}(\gamma^2 I_p - L_kP_kL_k^{\mathrm{T}})^{-1}L_kP_k \geq 0\end{aligned}$$

Then we get

$$\begin{aligned}\gamma^{-2}\bar{\Omega}_k &= R_k - C_kP_k\Sigma_k^{-1}C_k^{\mathrm{T}} \\ &= R_k - C_k\widetilde{P}_k(I_n + C_k^{\mathrm{T}}R_k^{-1}C_k\widetilde{P}_k)^{-1}C_k^{\mathrm{T}} \\ &= (R_k + C_k\widetilde{P}_kC_k^{\mathrm{T}})^{-1} > 0\end{aligned}$$

There also exists a matrix $\bar{P}_k = P_k(I_n + C_k^{\mathrm{T}}R_k^{-1}C_kP_k)^{-1}$ since $P_k \geq 0$. Thus we obtain

$$\bar{P}_k = P_k - P_kC_k^{\mathrm{T}}(R_k + C_kP_kC_k^{\mathrm{T}})^{-1}C_kP_k \leq P_k$$

Furthermore, since $\gamma^2 I_p - L_kP_kL_k^{\mathrm{T}} > 0$, we get

$$\gamma^2 I_p - L_k\bar{P}_kL_k^{\mathrm{T}} \geq \gamma^2 I_p - L_kP_kL_k^{\mathrm{T}} > 0 \tag{5.3.31}$$

We thus get $P_{k+1} \geq 0$ from Lemma 5.1. ■

In summary, the following theorem gives a necessary condition for the existence of the minimax prediction problem.

Theorem 5.2: *For the minimax prediction problem (5.2.9), a necessary condition for the existence of a unique solution is that the RDE (5.3.15) has a positive semi-definite solution P_k such that $\gamma^2 I_p - L_k P_k L_k^{\mathrm{T}} > 0$ for all $k \in [0, N]$. Then the optimal estimate $\widehat{z}_k^*$ is given by (5.3.28)–(5.3.30).*

Remark 5.2: Suppose that P_k satisfies the conditions in Theorems 5.1 and 5.2. Then, P_k is positive definite if A_k is nonsingular and/or B_k has full row rank. Thus, for simplicity of discussion, we hereafter assume that A_k is nonsingular for all k.

4. Sufficient Conditions

In this section, we show that the necessary conditions in Theorems 5.1 and 5.2 are also sufficient conditions in the case where A_k is nonsingular. Similarly to the reference [12], we can prove the sufficient conditions by completing the square argument.

In this section, we assume that $P_k > 0$ and $\gamma^2 I_p - L_k \bar{P}_k L_k^{\mathrm{T}} > 0$ hold for all $k \in [0, N]$. From (5.3.31), this assumption is valid in both filtering and prediction cases.

Lemma 5.3: *Suppose that the RDE (5.3.15) has a positive definite solution P_k such that $\gamma^2 I_p - L_k \bar{P}_k L_k^{\mathrm{T}} > 0$. Then there exists a positive definite symmetric matrix X_k satisfying*

$$X_k = A_k^{\mathrm{T}} X_{k+1} A_k + A_k^{\mathrm{T}} X_{k+1} B_k V_k^{-1} B_k^{\mathrm{T}} X_{k+1} A_k$$
$$+\gamma^{-2} L_k^{\mathrm{T}} L_k - C_k^{\mathrm{T}} R_k^{-1} C_k \qquad (5.4.1)$$

$$V_k = I_m - B_k^{\mathrm{T}} X_{k+1} B_k > 0 \qquad (5.4.2)$$

Proof: We first define $X_k = P_k^{-1}$, so that $P_k > 0$ implies $X_k > 0$. Moreover, since A_k and P_k are invertible, X_k satisfies (5.4.1) by the matrix inversion lemma.

Furthermore, since $P_k > 0$ and $\Omega_k^{-1} = \gamma^2 I_p - L_k \bar{P}_k L_k^{\mathrm{T}} > 0$, we get

$$\Xi_k = P_k \Sigma_k^{-1} = \bar{P}_k + \gamma^{-2} \bar{P}_k L_k^{\mathrm{T}} \Omega_k L_k \bar{P}_k > 0 \qquad (5.4.3)$$

Thus we see from (5.3.15) that $X_{k+1}^{-1} - B_k B_k^{\mathrm{T}} > 0$ holds. This implies (5.4.2). ■

We define $\widehat{x}_k$ by (5.3.16) and $\widetilde{x}_k = x_k - \widehat{x}_k$. Then, from (5.2.1) and (5.3.16), we get

$$\widetilde{x}_{k+1} = A_k \widetilde{x}_k + B_k w_k - A_k \Xi_k (C_k^{\mathrm{T}} R_k^{-1} \bar{y}_k - \gamma^{-2} L_k^{\mathrm{T}} \bar{z}_k) \qquad (5.4.4)$$

It thus follows from (5.3.15),(5.4.1) and (5.4.4) that

$$\begin{aligned}
&\widetilde{x}_{k+1}^{\mathrm{T}} X_{k+1} \widetilde{x}_{k+1} - \widetilde{x}_k^{\mathrm{T}} X_k \widetilde{x}_k \\
&\quad = \|w_k\|^2 + \|y_k - C_k x_k\|^2_{R_k^{-1}} - \gamma^{-2}\|z_k - \widehat{z}_k\|^2 \\
&\qquad - \left\| w_k - V_k^{-1} B_k X_{k+1} A_k \{\widetilde{x}_k - \Xi_k (C_k^{\mathrm{T}} R_k^{-1} \bar{y}_k - L_k^{\mathrm{T}} \bar{z}_k)\} \right\|^2_{V_k} \\
&\qquad + \gamma^{-2} \begin{bmatrix} \bar{z}_k \\ \bar{y}_k \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Omega_k & -L_k \Xi_k C_k^{\mathrm{T}} R_k^{-1} \\ -R_k^{-1} C_k \Xi_k L_k^{\mathrm{T}} & -R_k^{-1} \bar{\Omega}_k R_k^{-1} \end{bmatrix} \begin{bmatrix} \bar{z}_k \\ \bar{y}_k \end{bmatrix}
\end{aligned} \tag{5.4.5}$$

Furthermore, it is easy to verify that

$$\sum_{k=0}^{N} (\widetilde{x}_{k+1}^{\mathrm{T}} X_{k+1} \widetilde{x}_{k+1} - \widetilde{x}_k^{\mathrm{T}} X_k \widetilde{x}_k) = \widetilde{x}_{N+1}^{\mathrm{T}} X_{N+1} \widetilde{x}_{N+1} - \widetilde{x}_0^{\mathrm{T}} \Pi^{-1} \widetilde{x}_0$$

Hence, we obtain

$$\begin{aligned}
J(\widehat{z}; x_0, w, v) &= -\gamma^2 \sum_{k=0}^{N} \|w_k - w_k^*\|^2_{V_k} - \gamma^2 \widetilde{x}_{N+1}^{\mathrm{T}} X_{N+1} \widetilde{x}_{N+1} \\
&\quad + \sum_{k=0}^{N} \begin{bmatrix} \bar{z}_k \\ \bar{y}_k \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Omega_k & -L_k \Xi_k C_k^{\mathrm{T}} R_k^{-1} \\ -R_k^{-1} C_k \Xi_k L_k^{\mathrm{T}} & -R_k^{-1} \bar{\Omega}_k R_k^{-1} \end{bmatrix} \begin{bmatrix} \bar{z}_k \\ \bar{y}_k \end{bmatrix}
\end{aligned} \tag{5.4.6}$$

where

$$w_k^* = V_k^{-1} B_k X_{k+1} A_k \{\widetilde{x}_k - \Xi_k (C_k^{\mathrm{T}} R_k^{-1} \bar{y}_k - L_k^{\mathrm{T}} \bar{z}_k)\} \tag{5.4.7}$$

Since $V_k > 0$ for all $k \in [0, N]$, the worst case disturbance is uniquely determined by (5.4.7). Moreover, the next lemma holds for w_k^* of (5.4.7).

Lemma 5.4: *Let w_k^* be defined by (5.4.7), and define $\lambda_k = P_k^{-1} \widetilde{x}_k$. If we take $w_k = w_k^*$, then λ_k satisfies (5.3.14).*

Proof: From (5.4.4) and (5.4.7), we get

$$w_k^* = V_k^{-1} B_k X_{k+1} (\widetilde{x}_{k+1} - B_k w_k^*)$$

Noting (5.4.2), we solve the above equation to get

$$\begin{aligned}
w_k^* &= (I_m + V_k^{-1} B_k^{\mathrm{T}} X_{k+1} B_k)^{-1} V_k^{-1} B_k^{\mathrm{T}} X_{k+1} \widetilde{x}_{k+1} \\
&= B_k^{\mathrm{T}} X_{k+1} \widetilde{x}_{k+1}
\end{aligned}$$

Since $X_k = P_k^{-1}$ and $\lambda_k = P_k^{-1}\tilde{x}_k$, we get $w_k^* = B_k^{\mathrm{T}}\lambda_{k+1}$. By substituting this into (5.4.4), we get

$$(P_{k+1} - B_k^{\mathrm{T}}B_k)\lambda_{k+1} = A_kP_k\lambda_k - A_kP_k\Sigma_k^{-1}(C_k^{\mathrm{T}}R_k^{-1}\bar{y}_k - \gamma^{-2}L_k^{\mathrm{T}}\bar{z}_k)$$

From (5.3.15), pre-multiplying by $(A_kP_k\Sigma_k^{-1})^{-1}$ yields

$$A_k^{\mathrm{T}}\lambda_{k+1} = \Sigma_k\lambda_k - C_k^{\mathrm{T}}R_k^{-1}\bar{y}_k + \gamma^{-2}L_k^{\mathrm{T}}\bar{z}_k$$

■

Since $\bar{z}_k$ and $\bar{y}_k$ are independent of x_k, $\tilde{x}_{N+1} = 0$ (equivalently, $\lambda_{N+1} = 0$) holds for the optimal initial state x_0^*. Note that x_0^* can be uniquely obtained by calculating (5.3.14) backwards. Thus we get

$$J(\hat{z}; x_0^*, w^*, v) = \sum_{k=0}^{N} \begin{bmatrix} \bar{z}_k \\ \bar{y}_k \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Omega_k & -L_k\Xi_kC_k^{\mathrm{T}}R_k^{-1} \\ -R_k^{-1}C_k\Xi_kL_k^{\mathrm{T}} & -R_k^{-1}\bar{\Omega}_kR_k^{-1} \end{bmatrix} \begin{bmatrix} \bar{z}_k \\ \bar{y}_k \end{bmatrix}$$

Tracing back the discussions in the subsection 3.2 for the above equation, we obtain the following theorems.

Theorem 5.3: *Suppose that the RDE (5.3.15) has a positive definite solution P_k and $\gamma^2I_p - L_k\bar{P}_kL_k^{\mathrm{T}} > 0$ holds for all $k \in [0, N]$. Then the minimax problem (5.2.8) has a unique optimal solution.*

Theorem 5.4: *Suppose that the RDE (5.3.15) has a positive definite solution P_k and $\gamma^2I_p - L_kP_kL_k^{\mathrm{T}} > 0$ holds for all $k \in [0, N]$. Then the minimax problem (5.2.9) has a unique optimal solution.*

5. Relation to $\mathbf{H}_\infty$ Filtering Problem

We next show that if the minimax problem (5.2.6) (respectively, (5.2.7)) has a unique solution, then the optimal estimate $\hat{z}_k^*$ satisfies the $\mathbf{H}_\infty$ error bound (5.2.5).

Theorem 5.5: *Suppose that the RDE (5.3.15) has a positive definite solution P_k for all $k \in [0, N]$, and*

$$\gamma^2I_p - L_k\bar{P}_kL_k^{\mathrm{T}} > 0 \quad \forall k \in [0, N]$$

Then the filter of (5.3.20)–(5.3.23) achieves the $\mathbf{H}_\infty$ error bound (5.2.5).

Proof: Let us define $\widehat{z}_k^*$ by (5.3.20)–(5.3.23). Then we see from (5.4.6) and (5.3.19) that

$$J(\widehat{z}^*; x_0, w, v) = -\gamma^2\{\sum_{k=0}^{N} \|w_k - w_k^*\|_{V_k}^2 + \sum_{k=0}^{N} \bar{y}_k^{\mathrm{T}}(R_k + C_k P_k C_k^{\mathrm{T}})^{-1}\bar{y}_k\}$$
$$-\gamma^2 \tilde{x}_{N+1}^{\mathrm{T}} X_{N+1} \tilde{x}_{N+1} \le 0$$

Hence the optimal policies for $w_k, \bar{y}_k$ and $\tilde{x}_{N+1}$ which maximize $J(\widehat{z}^*; x_0, w, v)$ are given by

$$w_k = w_k^*, \quad \bar{y}_k = 0, \quad \tilde{x}_{N+1} = 0$$

Using $\bar{y}_k = 0, \tilde{x}_{N+1} = 0$ and the definition of $\widehat{z}_k^*$, (5.3.14) reduces to

$$A_k^{\mathrm{T}} \lambda_{k+1} = \Sigma_k \lambda_k, \quad \lambda_{N+1} = 0$$

This implies $\lambda_k = 0$, $\tilde{x}_k = 0$ for all $k \in [0, N]$. It then follows that

$$v_k^* = -D_k^{-1} C_k \tilde{x}_k = 0$$
$$w_k^* = B_k^{\mathrm{T}} \lambda_{k+1} = 0$$
$$x_0^* = \widehat{x}_0 + \Pi \lambda_0 = \bar{x}_0$$

Hence we obtain

$$\sum_{k=0}^{N} (\|w_k^*\|^2 + \|v_k^*\|_{R_k^{-1}}^2) + \|x_0^* - \bar{x}_0\|_{\Pi^{-1}}^2 = 0$$

Consequently, (5.2.6) (equivalently (5.2.5)) holds for the filter (5.3.20)–(5.3.23). ∎

A similar result is obtained for the one-step prediction problem.

Theorem 5.6: *Suppose that the RDE (5.3.15) has a positive definite solution P_k for all $k \in [0, N]$, and*

$$\gamma^2 I_p - L_k P_k L_k^{\mathrm{T}} > 0 \quad \forall k \in [0, N]$$

Then the one-step predictor (5.3.28)–(5.3.30) achieves the $\mathbf{H}_\infty$ error bound (5.2.5).

Since the filter (5.3.20)–(5.3.23) and the predictor (5.3.28)–(5.3.30) satisfy the $\mathbf{H}_\infty$ error bound. They are referred to as an $\mathbf{H}_\infty$ filter and an $\mathbf{H}_\infty$ predictor, respectively. Note also that, as γ tends to infinity, the filter of (5.3.20)–(5.3.23) reduces to the Kalman filter.

Therefore, the filter of (5.3.20)–(5.3.23) is called the central $\mathbf{H}_\infty$ filter. This definition of the central $\mathbf{H}_\infty$ filter is consistent with the definition in Chapter 2.

Remark 5.3: Fujita *et al.* [12] gave a similar result for filtering case under the condition of $P_k > 0$ and $\Xi_k > 0$. The equivalence of $\Xi_k > 0$ and $\gamma^2 I_p - L_k \bar{P}_k L_k^{\mathrm{T}} > 0$ is easily shown using the definition of Ξ_k.

Remark 5.4: If the initial state is exactly known a priori (i.e. $x_0 = \bar{x}_0$), then the $\mathbf{H}_\infty$ error bound and cost function J become

$$\sup_{w,v} \frac{\sum_{k=0}^{N} \|z_k - \hat{z}_k\|^2}{\sum_{k=0}^{N}(\|w_k\|^2 + \|v_k\|^2)} < \gamma^2$$

$$J(\hat{z}; w, v) = \sum_{k=0}^{N} \|z_k - \hat{z}_k\|^2 - \gamma^2 \sum_{k=0}^{N}(\|w_k\|^2 + \|v_k\|^2)$$

In this case, the solutions to the minimax problems are irrelevant to the weighting matrix Π, and we have $P_0 = 0$.

6. Concluding Remarks

In this chapter, we have shown that the solutions to the minimax filtering and predictions problems are given by the central $\mathbf{H}_\infty$ filter and $\mathbf{H}_\infty$ one-step predictor, respectively. Furthermore, in deducing the minimax solutions, we have derived the worst noise and disturbances in the sense that they maximizes the cost function (5.2.4), or equivalently they maximizes the energy gain between the estimation errors and the noise disturbances.

In the infinite-horizon time-varying case, in addition to the existence of a solution to the $\mathbf{H}_\infty$ RDE, it is required for existence of a solution to the $\mathbf{H}_\infty$ filtering problem that the Riccati solution P_k is an stabilizing solution, that is, the autonomous system $\xi_{k+1} = F_k \Sigma_k^{-\mathrm{T}} \xi_k$ is exponentially stable (see, e.g. [36] for the continuous-time case).

Appendix 5.1: Reformulation of Filtering Problem

We here reduce the filtering problem for (2.2.1),(2.2.2) to the problem for (5.2.1),(5.2.2) under the assumption that D has full row rank.

We now define $D^{\#} = D^{\mathrm{T}} R^{-1}$ to obtain from (2.2.2)

$$0 = BD^{\#}(-y_k + Cx_k + Dd_k)$$

Subtracting this from (2.2.1) yields

$$x_{k+1} = (A - BD^{\#}C)x_k + BD^{\perp}d_k + BD^{\#}y_k \tag{A.5.1}$$

where $D^{\perp} = I_m - D^{\#}D$. By linearity, x_k is decomposed as $x_k = x_k^{(1)} + x_k^{(2)}$, where

$$x_{k+1}^{(1)} = (A - BD^{\#}C)x_{k+1}^{(1)} + BD^{\perp}d_k, \quad x_0^{(1)} = x_0 \tag{A.5.2}$$

$$x_{k+1}^{(2)} = (A - BD^{\#}C)x_{k+1}^{(2)} + BD^{\#}y_k, \quad x_0^{(2)} = 0 \tag{A.5.3}$$

Clearly, given the history of the measurements, $x_k^{(2)}$ is known exactly and the task of estimation becomes that of estimating $x_k^{(1)}$ only. Further, we introduce a new measurement

$$y_k' = y_k - Cx_k^{(2)} = Cx_k^{(1)} + Dd_k \tag{A.5.4}$$

Since D has full row rank, there exists an orthogonal matrix U such that $DU = [\,D' \;\; 0\,]$ with D' nonsingular. Accordingly, we partition U as $U = [\,U_1 \;\; U_2\,]$ and define $\begin{bmatrix} v_k \\ w_k \end{bmatrix} = U^{\mathrm{T}}d_k = \begin{bmatrix} U_1^{\mathrm{T}}d_k \\ U_2^{\mathrm{T}}d_k \end{bmatrix}$. Then, (A.5.2) and (A.5.4) reduce to

$$x_{k+1}^{(1)} = (A - BD^{\#}C)x_k^{(1)} + BU_2w_k \tag{A.5.5}$$

$$y_k' = Cx_k^{(1)} + D'v_k \tag{A.5.6}$$

Since $\|d_k\|^2 = \|v_k\|^2 + \|w_k\|^2$, and since the estimation error is only due to the error in estimating $x_k^{(1)}$, the filtering problem for (2.2.1),(2.2.2) reduces to that for (5.2.1),(5.2.2) by redefining as follows:

$$\begin{array}{ccc} x_k^{(1)} & \longrightarrow & x_k \\ y_k' & \longrightarrow & y_k \\ A - BD^{\#}C & \longrightarrow & A \\ BU_2 & \longrightarrow & B \\ D' & \longrightarrow & D \end{array}$$

Appendix 5.2: Minimax Fixed-Interval Smoothing Problem

The minimax smoother which minimizes $J(\hat{z}; x_0^*, w^*, v)$ is obtained by taking $\hat{z}_k = L_k x_k^*$ in (5.3.11) since all the measurements $\{y_0, \cdots, y_N\}$ are available for the estimation at time $k \in [0, N]$. Thus the minimax smoother is given by

$$\hat{z}_k^* = L_k x_k^* \tag{A.5.7}$$

$$\begin{bmatrix} I_n & -B_k B_k^{\mathrm{T}} \\ 0 & A_k^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x_{k+1}^* \\ \lambda_{k+1}^* \end{bmatrix} = \begin{bmatrix} A_k & 0 \\ C_k^{\mathrm{T}} R_k^{-1} C_k & I_n \end{bmatrix} \begin{bmatrix} x_k^* \\ \lambda_k^* \end{bmatrix} + \begin{bmatrix} 0 \\ -C_k^{\mathrm{T}} R_k^{-1} y_k \end{bmatrix}, \quad \begin{cases} x_0^* = \bar{x}_0 + \Pi \lambda_0^* \\ \lambda_{N+1}^* = 0 \end{cases} \tag{A.5.8}$$

We see from this equation that the smoothed estimate $\hat{z}_k$ in the minimax smoothing is independent of γ and L_k. This feature makes the $\mathbf{H}_\infty$ smoother identical to the $\mathbf{H}_2$ optimal smoother. A necessary and sufficient condition for the $\mathbf{H}_\infty$ optimality of the smoother was given by Nagpal and Khargonekar [36] and Basar [2].

Theorem A.5.1 [36]: *A necessary and sufficient condition for the smoother (A.5.7),(A.5.8) to satisfy the* $\mathbf{H}_\infty$ *error bound is that there exists a matrix* X_k *satisfying (5.4.1) with* $X_{N+1} = 0$ *and* $X_0 < \Pi$.

Chapter 6

Performance of Central $\mathbf{H}_\infty$ Filter, $\mathbf{H}_\infty$ Riccati Difference Equation and $\mathbf{H}_\infty$ Fixed-Lag Smoothing Problem

1. Introduction

As shown in the previous chapters, the $\mathbf{H}_\infty$ filtering problem has been solved from various viewpoints [12],[19],[20], [52],[53],[55]. At present, however, the performance of the $\mathbf{H}_\infty$ filter has received much less attention. Thus, in this chapter, we will study the performance of the central $\mathbf{H}_\infty$ filter based on Riccati difference equations. It is well known that, as the prescribed $\mathbf{H}_\infty$ bound γ tends to ∞, the $\mathbf{H}_\infty$ filtering problem reduces to the $\mathbf{H}_2$-optimal filtering problem. Kalman filter offers the optimal state estimates in the least-squares error sense when the disturbance is zero mean white noise and its covariance is exactly known. Thus, we first consider the performance in the case when the underlying disturbance is zero mean white noise by comparing the $\mathbf{H}_\infty$ and $\mathbf{H}_2$ (Kalman filtering) RDEs. Next, we clarify the the relationship between γ and the performance of the central $\mathbf{H}_\infty$ filter based on the monotonicity of the $\mathbf{H}_\infty$ RDE.

Next, for a time-invariant system, we will show that, under a certain condition, the solution of the $\mathbf{H}_\infty$ RDE converges to a stabilizing solution of the corresponding $\mathbf{H}_\infty$

ARE. This result gives a connection between the finite and infinite horizon $\mathbf{H}_\infty$ filtering problems.

Furthermore, for the case where a fixed time lag is allowed between measurement and estimation, the state estimator is termed a fixed-lag smoother. As well-known, there are many applications particularly to communication systems where a delay sufficient to yield a useful improvement in estimation from smoothing is acceptable [26],[35]. Based on the precedent results on the $\mathbf{H}_\infty$ filtering problem, we will derive a solution to the $\mathbf{H}_\infty$ fixed-lag smoothing problem.

2. Performance Analysis of Central $\mathbf{H}_\infty$ Filter

2.1 Finite-horizon $\mathbf{H}_\infty$ filtering problem

We now briefly review the result on the finite-horizon $\mathbf{H}_\infty$ filtering problem. We again consider the system described by

$$x_{k+1} = A_k x_k + B_k w_k \tag{6.2.1}$$

$$y_k = C_k x_k + D_k v_k \tag{6.2.2}$$

$$z_k = L_k x_k \tag{6.2.3}$$

where $x_k \in \mathbf{R}^n$, $y_k \in \mathbf{R}^q$ and $z_k \in \mathbf{R}^p$ are the state vector, measurement and the vector to to be estimated. The exogenous signals $w_k \in \mathbf{R}^m$ and $v_k \in \mathbf{R}^\ell$ are the process disturbance and the measurement noise, respectively. Moreover, we assume that $R_k := D_k D_k^{\mathrm{T}} > 0$ holds for any k.

The finite-horizon $\mathbf{H}_\infty$ filtering problem is to find estimates of z_k and x_k based on the measurement set $\{y_0, \cdots, y_k\}$ such that

$$\sup_{w,v,x_0} \frac{\sum_{k=0}^{N} \|z_k - \hat{z}_k\|^2}{\sum_{k=0}^{N}(\|w_k\|^2 + \|v_k\|^2) + \|x_0 - \bar{x}_0\|^2_{\Pi^{-1}}} < \gamma^2 \tag{6.2.4}$$

where $\hat{z}_k$ is the estimate of z_k, and $\bar{x}_0$ is a priori estimate of the initial state x_0. Also, Π is a positive definite weighting matrix which represents the uncertainty of the initial state. As shown in Chapter 5, the central $\mathbf{H}_\infty$ filter which achieves the above $\mathbf{H}_\infty$ bound is given by

$$\widehat{x}_{k/k} = \widehat{x}_{k/k-1} + K_k(y_k - C_k\widehat{x}_{k/k-1}) \tag{6.2.5}$$

$$\widehat{x}_{k+1/k} = A_k\widehat{x}_{k/k}, \quad \widehat{x}_{0/-1} = \bar{x}_0 \tag{6.2.6}$$

$$\widehat{z}_k = L_k\widehat{x}_{k/k} \tag{6.2.7}$$

$$K_k = P_kC_k^{\mathrm{T}}(R_k + C_kP_kC_k^{\mathrm{T}})^{-1} \tag{6.2.8}$$

where P_k satisfies the RDE

$$P_{k+1} = A_kP_k\{I_n + (C_k^{\mathrm{T}}R_k^{-1}C_k - \gamma^{-2}L_k^{\mathrm{T}}L_k)P_k\}^{-1}A_k^{\mathrm{T}} + B_kB_k^{\mathrm{T}}, \quad P_0 = \Pi \tag{6.2.9}$$

and

$$V_k := \gamma^2I_p - L_kP_k(I_n + C_k^{\mathrm{T}}R_k^{-1}C_kP_k)^{-1}L_k^{\mathrm{T}} > 0 \tag{6.2.10}$$

2.2 Estimation error covariance

We define

$$J = \sum_{k=0}^{N}\|z_k - \widehat{z}_k\|^2 - \gamma^2(\sum_{k=0}^{N}\|w_k\|^2 + \sum_{k=0}^{N}\|v_k\|^2 + \|x_0 - \bar{x}_0\|^2_{\Pi^{-1}}) \tag{6.2.11}$$

We see from Theorems 5.1 and 5.3 that the filter of (6.2.5)–(6.2.8) is the optimal minimizing policy of the minimax problem:

$$\max_{y_N}(\min_{\widehat{z}_N}(\max_{w_N}\cdots\max_{y_k}(\min_{\widehat{z}_k}(\max_{w_k} \cdots\max_{y_0}(\min_{\widehat{z}_k}(\max_{w_0,x_0} J))\cdots))\cdots)) \tag{6.2.12}$$

As γ tends to infinity, the second term in $J(\widehat{z}; x_0, w, v)$ becomes dominant and the minimax problem reduces to the minimization problem:

$$\min_{w,x}\left\{\sum_{k=0}^{N}(\|w_k\|^2 + \|y_k - C_kx_k\|^2_{R_k^{-1}}) + \|x_0 - \bar{x}_0\|^2_{\Pi^{-1}}\right\}$$

As well-known, this minimization problem is equivalent to the minimum-variance estimation or least-squares estimation problem where x_0 is generated by the Gaussian distribution $\mathcal{N}(\bar{x}_0, \Pi)$ and where w_k and v_k are the Gaussian white noise processes such that

$$E\{w_k\} = 0, \quad E\{v_k\} = 0 \tag{6.2.13}$$

$$E\left\{\begin{bmatrix} w_k \\ v_k \end{bmatrix}\begin{bmatrix} w_\tau \\ v_\tau \end{bmatrix}^{\mathrm{T}}\right\} = \begin{bmatrix} I_m & 0 \\ 0 & I_\ell \end{bmatrix}\delta_{k\tau} \tag{6.2.14}$$

Among all causal state estimators, the optimal solution to this problem is given by the Kalman filter:

$$\widehat{x}_{k/k} = \widehat{x}_{k/k-1} + K'_k(y_k - C_k\widehat{x}_{k/k-1}) \tag{6.2.15}$$

$$\widehat{x}_{k+1/k} = A_k\widehat{x}_{k/k}, \quad \widehat{x}_{0/-1} = \bar{x}_0 \tag{6.2.16}$$

$$K'_k = P'_kC_k^{\mathrm{T}}(R_k + C_kP'_kC_k^{\mathrm{T}})^{-1} \tag{6.2.17}$$

where the matrix P'_k is the optimal one-step prediction error covariance matrix

$$P'_k = E\{(x_k - \widehat{x}_{k/k-1})(x_k - \widehat{x}_{k/k-1})^{\mathrm{T}}\}$$

and satisfies the following RDE.

$$P'_{k+1} = A_kP'_kA_k^{\mathrm{T}} - A_kP'_kC_k^{\mathrm{T}}(R_k + C_kP'_kC_k^{\mathrm{T}})^{-1}C_kP'_kA_k^{\mathrm{T}} + B_kB_k^{\mathrm{T}}, \quad P'_0 = \Pi \tag{6.2.18}$$

It follows from the above observation that the $\mathbf{H}_\infty$ filter is a modified version of Kalman filter by using the parameter γ. Hence, it is very important to compare the performances of the $\mathbf{H}_\infty$ and Kalman filters when w_k, v_k and x_0 are given by the Gaussian white noise processes. In the following of this chapter, we assume that there exists a positive semi-definite solution P_k to the RDE (6.2.9) satisfying $V_k > 0$ exists for all $k \in [0, N]$.

Theorem 6.1: *Suppose that $x_0 \sim \mathcal{N}(\bar{x}_0, \Pi)$ and w_k, v_k are the zero mean Gaussian white noises with unit covariance matrices. Define*

$$Q_k = E\{(x_k - \widehat{x}_{k/k-1})(x_k - \widehat{x}_{k/k-1})^{\mathrm{T}}\}$$

for the $\mathbf{H}_\infty$ filter of (6.2.5)–(6.2.8). Then $P_k \geq Q_k \geq P'_k$ holds for all $k \in [0, N]$.

Proof: We define

$$F_k = A_kK_k = A_kP_kC_k^{\mathrm{T}}(R_k + C_kP_kC_k^{\mathrm{T}})^{-1}$$

$$F'_k = A_kK'_k = A_kP'_kC_k^{\mathrm{T}}(R_k + C_kP'_kC_k^{\mathrm{T}})^{-1}$$

From (6.2.1)–(6.2.3), (6.2.5) and (6.2.6), the dynamics of the estimation error $\widetilde{x}_k := x_k - \widehat{x}_{k/k-1}$ is described by

$$\widetilde{x}_{k+1} = (A_k - F_kC_k)\widetilde{x}_k + B_kw_k - F_kD_kv_k, \quad \widetilde{x}_0 = x_0 - \bar{x}_0$$

It follows that

$$Q_{k+1} = (A_k - F_kC_k)Q_k(A_k - F_kC_k)^{\mathrm{T}} + F_kR_kF_k^{\mathrm{T}} + B_kB_k^{\mathrm{T}}, \quad Q_0 = \Pi \tag{6.2.19}$$

Also, after some simple calculations, the RDE (6.2.9) reduces to

$$\begin{aligned} P_{k+1} &= (A_k - F_kC_k)P_k(A_k - F_kC_k)^{\mathrm{T}} + F_kR_kF_k^{\mathrm{T}} + B_kB_k^{\mathrm{T}} \\ &+ A_k\bar{P}_kL_k^{\mathrm{T}}(\gamma^2 I_p - L_k\bar{P}_kL_k^{\mathrm{T}})^{-1}L_k\bar{P}_kA_k^{\mathrm{T}}, \quad P_0 = \Pi \end{aligned} \tag{6.2.20}$$

where $\bar{P}_k := P_k(I_n + C_k^{\mathrm{T}}R_k^{-1}C_k)^{-1} \geq 0$. Subtracting (6.2.19) from (6.2.20) yields

$$\begin{aligned} P_{k+1} - Q_{k+1} &= (A_k - F_kC_k)(P_k - Q_k)(A_k - F_kC_k)^{\mathrm{T}} \\ &+ A_k\bar{P}_kL_k^{\mathrm{T}}(\gamma^2 I_p - L_k\bar{P}_kL_k^{\mathrm{T}})^{-1}L_k\bar{P}_kA_k^{\mathrm{T}}, \quad P_0 - Q_0 = 0 \end{aligned}$$

Since $V_k = \gamma^2 I_p - L_k\bar{P}_kL_k^{\mathrm{T}} > 0$ holds for all $k \in [0, N]$, we get $P_k - Q_k \geq 0$ for all $k \in [0, N]$ by induction.

Next we prove $Q_k \geq P_k'$. It is easily verified that

$$\begin{aligned} P_{k+1}' &= (A_k - F_k'C_k)P_k'(A_k - F_k'C_k)^{\mathrm{T}} + F_k'R_kF_k'^{\mathrm{T}} + B_kB_k^{\mathrm{T}} \\ &= (A_k - F_kC_k)P_k'(A_k - F_kC_k)^{\mathrm{T}} + B_kB_k^{\mathrm{T}} + F_kR_kF_k^{\mathrm{T}} \\ &-(F_k - F_k')R_k(F_k - F_k')^{\mathrm{T}}, \quad P_0' = \Pi \end{aligned} \tag{6.2.21}$$

Subtracting (6.2.19) from this yields

$$P_{k+1}' - Q_{k+1} = (A_k - F_kC_k)(P_k' - Q_k)(A_k - F_kC_k)^{\mathrm{T}} - (F_k - F_k')R_k(F_k - F_k')^{\mathrm{T}}$$

where $P_0' - Q_0 = 0$. Since $R_k > 0$, we get $P_k' - Q_k \leq 0$ for all $k \in [0, N]$ by induction. ■

We now define

$$\begin{aligned} \bar{P}_k &= P_k(I_n + C_k^{\mathrm{T}}R_k^{-1}C_kP_k)^{-1} \\ \bar{P}_k' &= P_k'(I_n + C_k^{\mathrm{T}}R_k^{-1}C_kP_k')^{-1} \end{aligned}$$

Then we have the following lemma.

Lemma 6.1: *For symmetric matrices P_k and P_k', if $P_k \geq P_k' \geq 0$ holds, then $\bar{P}_k \geq \bar{P}_k' \geq 0$ holds.*

Proof: We easily see that

$$\bar{P}_k = (I_n - K_kC_k)P_k(I_n - K_kC_k)^{\mathrm{T}} + K_kR_kK_k^{\mathrm{T}} \geq 0 \tag{6.2.22}$$

$$\bar{P}'_k = (I_n - K'_kC_k)P'_k(I_n - K'_kC_k)^{\mathrm{T}} + K'_kR_kK'^{\mathrm{T}}_k \geq 0 \tag{6.2.23}$$

where K_k and K'_k are defined by (6.2.8) and (6.2.17), respectively. We also rewrite (6.2.23) as

$$\begin{aligned}\bar{P}'_k = (I_n - K_kC_k)P'_k(I_n - K_kC_k)^{\mathrm{T}} + K_kR_kK_k^{\mathrm{T}} \\ -(K'_k - K_k)(R_k + C_kP'_kC_k^{\mathrm{T}})(K'_k - K_k)^{\mathrm{T}}\end{aligned} \tag{6.2.24}$$

Subtracting this from (6.2.22) yields

$$\begin{aligned}\bar{P}_k - \bar{P}'_k = (I_n - K_kC_k)(P_k - P'_k)(I_n - K_kC_k)^{\mathrm{T}} \\ +(K'_k - K_k)(R_k + C_kP'_kC_k^{\mathrm{T}})(K'_k - K_k)^{\mathrm{T}}\end{aligned} \tag{6.2.25}$$

The right-hand side of the above equation is positive semi-definite since $P_k \geq P'_k \geq 0$ and $R_k > 0$. Thus we get $\bar{P}_k \geq \bar{P}'_k \geq 0$. ∎

The gain matrices K_k and K'_k can be expressed as

$$K_k = \bar{P}_kC_k^{\mathrm{T}}R_k^{-1}, \quad K'_k = \bar{P}'_kC_k^{\mathrm{T}}R_k^{-1}$$

Thus, from Theorem 6.1 and Lemma 6.1, we get $\|K_k\| \geq \|K'_k\|$. This implies that the $\mathbf{H}_\infty$ filter is more sensitive to $y_k - C_k\widehat{x}_{k/k-1}$ than Kalman filter. In the case where the measurement noise v_k is small, the estimate by the $\mathbf{H}_\infty$ filter converges to the actual state faster than Kalman filter.

2.3 Relationship between γ and $\mathbf{H}_\infty$ RDE

We define

$$\psi_k(P,\gamma) = P + PL_k^{\mathrm{T}}(\gamma^2I_p - L_kPL_k^{\mathrm{T}})^{-1}L_kP$$

Then we have the following lemma.

Lemma 6.2: *Assume that $P^{(1)} \geq P^{(2)} \geq 0$ and $\gamma^2I_p - L_kP^{(1)}L_k^{\mathrm{T}} > 0$ hold for a given $n \times n$ symmetric matrices $P^{(1)}$ and $P^{(2)}$. Then we obtain*

$$\psi_k(P^{(1)},\gamma) \geq \psi_k(P^{(2)},\gamma) \tag{6.2.26}$$

Proof: We define

$$M_k^{(i)} = \gamma P^{(i)} L_k^{\mathrm{T}} (\gamma^2 I_p - L_k P^{(i)} L_k^{\mathrm{T}})^{-1} \quad (i = 1, 2)$$
$$\varphi_k(P, M) = \left(I_n + \frac{ML_k}{\gamma}\right) P \left(I_n + \frac{ML_k}{\gamma}\right)^{\mathrm{T}} - MM^{\mathrm{T}}$$

It is clear that $\psi_k(P^{(i)}, \gamma) = \varphi_k(P^{(i)}, M_k^{(i)})$.

For any matrix $M \in \mathbf{R}^{n\times p}$, we get

$$\varphi_k(P^{(i)}, M_k^{(i)}) = \varphi_k(P^{(i)}, M) + \frac{1}{\gamma^2}(M_k^{(i)} - M)(\gamma^2 I_p - L_k P^{(i)} L_k^{\mathrm{T}})(M_k^{(i)} - M)^{\mathrm{T}} \quad (i = 1, 2)$$

It follows that

$$\begin{aligned}
\psi_k(P^{(1)}, \gamma) - \psi_k(P^{(2)}, \gamma) &= \varphi_k(P^{(1)}, M_k^{(1)}) - \varphi_k(P^{(2)}, M_k^{(2)}) \\
&= \varphi_k(P^{(1)}, M_k^{(2)}) + \frac{1}{\gamma^2}(M_k^{(1)} - M_k^{(2)})(\gamma^2 I_p - L_k P^{(1)} L_k^{\mathrm{T}})(M_k^{(1)} - M_k^{(2)})^{\mathrm{T}} \\
&\quad -\varphi_k(P^{(2)}, M_k^{(2)}) \\
&= \left(I_n + \frac{M_k^{(2)} L_k}{\gamma}\right)(P^{(1)} - P^{(2)})\left(I_n + \frac{M_k^{(2)} L_k}{\gamma}\right)^{\mathrm{T}} \\
&\quad + \frac{1}{\gamma^2}(M_k^{(1)} - M_k^{(2)})(\gamma^2 I_p - L_k P^{(1)} L_k^{\mathrm{T}})(M_k^{(1)} - M_k^{(2)})^{\mathrm{T}}
\end{aligned}$$

Since $P^{(1)} \geq P^{(2)}$ and $\gamma^2 I_p - L_k P^{(1)} L_k^{\mathrm{T}} > 0$, we obtain $\psi_k(P^{(1)}, \gamma) \geq \psi_k(P^{(2)}, \gamma)$. ■

Let $P^{(i)}$ denote the solution to the RDE (6.2.9) for given γ_i $(i = 1, 2)$. Then, $P_k^{(i)}$ satisfies

$$P_{k+1}^{(i)} = A_k P_k^{(i)} \{I_n + (C_k^{\mathrm{T}} R_k^{-1} C_k - \gamma^{-2} L_k^{\mathrm{T}} L_k) P_k^{(i)}\}^{-1} A_k^{\mathrm{T}} + B_k B_k^{\mathrm{T}},\ P_0^{(i)} = \Pi \quad (6.2.27)$$

We also define

$$\bar{P}_k^{(i)} = P_k^{(i)} (I_n + C_k^{\mathrm{T}} R_k^{-1} C_k P_k^{(i)})^{-1}$$

Theorem 6.2: *Suppose that $\gamma_1 \leq \gamma_2$, and that the RDE of (6.2.9) has positive semi-definite solutions $P_k^{(i)}$ $(i = 1, 2)$ such that $\gamma_i^2 I_p - L_k \bar{P}_k^{(i)} L_k^{\mathrm{T}} > 0$ for all $k \in [0, N]$. Then, $P_k^{(1)} \geq P_k^{(2)}$ and $\bar{P}_k^{(1)} \geq \bar{P}_k^{(2)}$ hold for all $k \in [0, N]$.*

Proof: Since $P_k^{(1)} \geq P_k^{(2)}$ implies $\bar{P}_k^{(1)} \geq \bar{P}_k^{(2)}$ by Lemma 6.1, it suffices to prove $P_k^{(1)} \geq P_k^{(2)}$.

For $k = 0$, it is obvious that $P_0^{(1)} = P_0^{(2)} = \Pi$.

We assume that $P_k^{(1)} \geq P_k^{(2)}$ holds for $k = 0, 1, \cdots, t$. Since $P_k^{(i)} \geq 0$, $\bar{P}_k^{(i)}$ $(i = 1, 2)$ are well-defined and positive semi-definite. We see from Lemma 6.1 that $\bar{P}_t^{(1)} \geq \bar{P}_t^{(2)}$. It thus follows from Lemma 6.2 that

$$\psi_t(\bar{P}_t^{(1)}, \gamma_1) \geq \psi_t(\bar{P}_t^{(1)}, \gamma_2) \geq \psi_t(\bar{P}_t^{(2)}, \gamma_2) \tag{6.2.28}$$

By simple calculations, (6.2.9) reduces to

$$P_{t+1}^{(i)} = A_t \psi_t(\bar{P}_t^{(i)}, \gamma_i) A_t^{\mathrm{T}} + B_t B_t^{\mathrm{T}} \tag{6.2.29}$$

Thus, we get

$$P_{t+1}^{(1)} - P_{t+1}^{(2)} = A_t \{\psi_t(\bar{P}_t^{(1)}, \gamma_1) - \psi_t(\bar{P}_t^{(2)}, \gamma_2)\} A_t^{\mathrm{T}} \geq 0 \tag{6.2.30}$$

As a result, we have shown by induction that $P_k^{(1)} \geq P_k^{(2)}$ holds for all $k \in [0, N]$. ■

The above theorem shows that the solution to the RDE (6.2.9) is monotonically non-increasing with respect to the parameter γ. By the discussion similar to the previous section, as γ becomes large, P_k and $E\{\|x_k - \widehat{x}_{k/k-1}\|^2\}$ become small, while the $\mathbf{H}_\infty$ filter gets less sensitive to $\nu_k := y_k - C_k \widehat{x}_{k/k-1}$. As a result, the parameter γ represents the trade off between the mean square error and the sensitivity to the estimation error ν_k.

3. Convergence of the Solution of $\mathbf{H}_\infty$ RDE

In this section, we consider the connection between the finite and infinite horizon $\mathbf{H}_\infty$ filtering problems by showing the convergence of the solution of the $\mathbf{H}_\infty$ RDE (6.2.9). We here assume that the coefficient matrices of the system (6.2.1)–(6.2.3) are constant. According to the assumptions (A1),(A2) in Chapter 2, we assume that (C, A) is detectable and that (A, B) has no uncontrollable modes on the unit circle.

Theorem 6.3: *Suppose that there exists a positive semi-definite stabilizing solution of the ARE*

$$P = AP\{I_n + (C^{\mathrm{T}} R^{-1} C - \gamma^{-2} L^{\mathrm{T}} L) P\}^{-1} A^{\mathrm{T}} + BB^{\mathrm{T}} \tag{6.3.1}$$

with $V := \gamma^2 I_p - L\bar{P}L^{\mathrm{T}} > 0$, where $\bar{P} = P(I_n + C^{\mathrm{T}} R^{-1} CP)^{-1}$. Then, the solution to the RDE (6.2.9) with $P_0 = 0$ converges to the stabilizing solution P as k tends to infinity.

Proof: We show that P_k is monotonically non-decreasing and bounded above, namely

$$P \geq \cdots \geq P_k \geq P_{k-1} \geq \cdots \geq P_1 \geq P_0 = 0 \tag{6.3.2}$$

For $k = 1$, it is obvious from $P_0 = 0$ that $P_1 = BB^{\mathrm{T}} \leq P$ holds.

For $k = t + 1$, we assume that

$$P \geq P_t \geq P_{t-1} \geq \cdots P_1 \geq P_0 = 0$$

We see from Lemma 6.1 that

$$\bar{P} \geq \bar{P}_t \geq \bar{P}_{t-1} \geq \cdots \bar{P}_1 \geq \bar{P}_0 = 0 \tag{6.3.3}$$

Hence, we get

$$0 < \gamma^2 I_p - L\bar{P}L^{\mathrm{T}} \leq \gamma^2 I_p - L\bar{P}_t L^{\mathrm{T}} \leq \gamma^2 I_p - L\bar{P}_{t-1} L^{\mathrm{T}}$$

It thus follows from Lemma 6.2 that

$$\psi(\bar{P}, \gamma) \geq \psi(\bar{P}_t, \gamma) \geq \psi(\bar{P}_{t-1}, \gamma) \geq 0$$

where $\psi(X, \gamma) = X + XL^{\mathrm{T}}(\gamma^2 I_p - LXL^{\mathrm{T}})^{-1}LX$. Since P_k and P are expressed as

$$P = A\psi(\bar{P}, \gamma)A^{\mathrm{T}} + BB^{\mathrm{T}}$$
$$P_{t+1} = A\psi(\bar{P}_t, \gamma)A^{\mathrm{T}} + BB^{\mathrm{T}}$$

the above inequality implies $P \geq P_{t+1} \geq P_t$. Thus, we obtain (6.3.2) by induction.

Furthermore, the stabilizing solution P is minimal among all the positive semi-definite solution of (6.3.1) with $V > 0$ [43]. Therefore, P_k converges to the stabilizing solution of (6.3.1). ■

It may be note that the ARE (6.3.1) can be rewritten as

$$P = APA^{\mathrm{T}} - AP\widehat{C}^{\mathrm{T}}(\widehat{R} + \widehat{C}P\widehat{C}^{\mathrm{T}})^{-1}\widehat{C}PA^{\mathrm{T}} + BB^{\mathrm{T}}$$
$$\widehat{C} = \begin{bmatrix} C \\ L \end{bmatrix}, \quad \widehat{R} = \begin{bmatrix} DD^{\mathrm{T}} & 0 \\ 0 & -\gamma^2 I_p \end{bmatrix}$$

Therefore, for the system of (6.2.1)–(6.2.3), the ARE (6.3.1) is identical to the ARE (2.4.6) in Theorem 2.1.

This theorem shows that if the infinite-horizon $\mathbf{H}_\infty$ filtering problem is solvable, then we can obtain a solution to this problem as the limit of the solution to the finite-horizon problem with $x_0 = \bar{x}_0$.

4. $\mathbf{H}_\infty$ Fixed-Lag Smoothing Problem

In this section, we derive a fixed-lag smoother which achieves the $\mathbf{H}_\infty$ error bound. We consider the system of (6.2.1)–(6.2.3) under the assumption that $x_{-i},\ \ i = 0, 1, \cdots, h$ are exactly known. The $\mathbf{H}_\infty$ fixed-lag smoothing problem is the problem of finding estimates $\widehat{x}_k$ and $\widehat{z}_k$ based on the measurement set $\{y_0, \cdots, y_{k+h}\}$ so that

$$\sup_{w,v} \frac{\sum_{k=0}^{N} \|z_{k-h} - \widehat{z}_{k-h}\|^2}{\sum_{k=0}^{N}(\|w_k\|^2 + \|v_k\|^2)} < \gamma^2 \tag{6.4.1}$$

where the constant $h \geq 0$ denotes the smoothing lag. The $\mathbf{H}_\infty$ fixed-lag smoothing problem can be easily solved by applying the usual $\mathbf{H}_\infty$ filtering algorithm to the augmented system:

$$\xi_{k+1} = \bar{A}_k \xi_k + \bar{B}_k w_k \tag{6.4.2}$$

$$y_k = \bar{C}_k \xi_k + D_k v_k \tag{6.4.3}$$

$$z_{k-h} = \bar{L}_k \xi_k \tag{6.4.4}$$

where

$$\xi_k = \begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_{k-h} \end{bmatrix}, \quad \bar{A}_k = \begin{bmatrix} A_k & 0 & \cdots & 0 \\ I_n & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & I_n & 0 \end{bmatrix}, \quad \bar{B}_k = \begin{bmatrix} B_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
$$\bar{C}_k = \begin{bmatrix} C_k & 0 & \cdots & 0 \end{bmatrix}, \quad \bar{L}_k = \begin{bmatrix} 0 & 0 & \cdots & L_{k-h} \end{bmatrix}$$

Theorem 6.4: *Suppose that $x_{-i},\ i = 0, 1, \cdots, h$ are known a priori. There exists an $\mathbf{H}_\infty$ fixed-lag smoother which achieves the $\mathbf{H}_\infty$ error bound (6.4.1) if and only if there exist matrices $P_{i,j/k}$, $\tilde{P}_{i,j/k}$ and $\Pi_{i,j/k}$ $(i, j = 0, 1, \cdots, h)$ such that*

$$P_{0,0/k+1} = A_k\bar{P}_{0,0/k}A_k^{\mathrm{T}} + A_k\Pi_{0,0/k}A_k^{\mathrm{T}} + B_kB_k^{\mathrm{T}} \tag{6.4.5a}$$

$$P_{0,j/k+1} = A_k\bar{P}_{0,j-1/k} + A_k\Pi_{0,j-1}, \quad j \neq 0 \tag{6.4.5b}$$

$$P_{i,j/k+1} = \bar{P}_{i-1,j-1/k} + \Pi_{i-1,j-1/k}, \quad ij \neq 0 \tag{6.4.5c}$$

$$\Pi_{i,j/k} = \bar{P}_{i,h}L_{k-h}^{\mathrm{T}}(\gamma^2 I_p - L_{k-h}\bar{P}_{h,h/k}L_{k-h}^{\mathrm{T}})^{-1}L_{k-h}\bar{P}_{h,j/k} \tag{6.4.5d}$$

$$\bar{P}_{i,j/k} = P_{i,j/k} - P_{i,0/k}C_k^{\mathrm{T}}(R_k + C_kP_{0,0/k}C_k^{\mathrm{T}})^{-1}C_k^{\mathrm{T}}P_{0,j/k} \tag{6.4.5e}$$

and

$$P_k := \begin{bmatrix} P_{0,0/k} & P_{0,1/k} & \cdots & P_{0,h/k} \\ P_{1,0/k} & P_{1,1/k} & \cdots & P_{1,h/k} \\ \vdots & \vdots & \ddots & \vdots \\ P_{h,0/k} & P_{h,1/k} & \cdots & P_{h,h/k} \end{bmatrix} \geq 0, \quad P_0 = 0 \tag{6.4.6}$$

$$\gamma^2 I_p - L_{k-h}\bar{P}_{h,h/k}L_{k-h}^{\mathrm{T}} > 0 \tag{6.4.7}$$

In this case, one of the $\mathbf{H}_\infty$ *fixed-lag smoother is given by*

$$\widehat{x}_{k-i/k} = \widehat{x}_{k-i/k-1} + K_{i/k}(y_k - C_k\widehat{x}_{k/k-1}) \tag{6.4.8a}$$

$$\widehat{x}_{k+1/k} = A_k\widehat{x}_{k/k} \tag{6.4.8b}$$

$$\widehat{z}_{k-h} = L_{k-h}\widehat{x}_{k-h/k} \tag{6.4.8c}$$

$$K_{i/k} = P_{i,0/k}C_k^{\mathrm{T}}(R_k + C_kP_{0,0/k}C_k^{\mathrm{T}})^{-1} \tag{6.4.8d}$$

where $\widehat{x}_{-i/-1} = x_{-i}, \quad i = 0, 1, \cdots, h.$

Proof: Since the $\mathbf{H}_\infty$ fixed-lag smoothing problem for (6.2.1)–(6.2.3) is equivalent to the $\mathbf{H}_\infty$ filtering problem for the augmented system (6.4.2)–(6.4.4), we easily see that an $\mathbf{H}_\infty$ fixed-lag smoother exists if and only if there exists a positive semi-definite solution to the RDE

$$P_{k+1} = \bar{A}_kP_k\Sigma_k^{-1}\bar{A}_k^{\mathrm{T}} + \bar{B}_k\bar{B}_k^{\mathrm{T}}, \quad P_0 = 0 \tag{6.4.9a}$$

$$\Sigma_k = I_{(h+1)n} + (\bar{C}_kR_k^{-1}C_k - \gamma^{-2}\bar{L}_k^{\mathrm{T}}\bar{L}_k)P_k \tag{6.4.9b}$$

with

$$\gamma^2 I_p - \bar{L}_k\bar{P}_k\bar{L}_k^{\mathrm{T}} > 0 \tag{6.4.10}$$

where $\bar{P}_k = P_k(I_{(h+1)n}+\bar{C}_k^{\mathrm{T}} R_k^{-1}\bar{C}_k P_k)^{-1}$. We now define $\Pi_k = \bar{P}_k\bar{L}_k^{\mathrm{T}}(\gamma^2 I_p - \bar{L}_k\bar{P}_k\bar{L}_k^{\mathrm{T}})^{-1}\bar{L}_k\bar{P}_k$, and partition X_k $(X = P, \bar{P}, \Pi)$ as

$$X_k := \begin{bmatrix} X_{0,0/k} & X_{0,1/k} & \cdots & X_{0,h/k} \\ X_{1,0/k} & X_{1,1/k} & \cdots & X_{1,h/k} \\ \vdots & \vdots & \ddots & \vdots \\ X_{h,0/k} & X_{h,1/k} & \cdots & X_{h,h/k} \end{bmatrix}$$

Then, by the definitions of $\bar{P}_k$ and Π_k, (6.4.5) immediately follows from the RDE (6.4.9). Also, the condition of (6.4.7) is equivalent to (6.4.10).

Futhermore, the central $\mathbf{H}_\infty$ filter for the augmented system is given by

$$\widehat{\xi}_{k/k} = \widehat{\xi}_{k/k-1} + K_k(y_k - \bar{C}_k\widehat{\xi}_{k/k-1}) \tag{6.4.11a}$$

$$\widehat{\xi}_{k+1/k} = \bar{A}_k\widehat{\xi}_{k/k}, \quad \widehat{\xi}_{0/-1} = \xi_0 \tag{6.4.11b}$$

$$\widehat{z}_{k-h} = \bar{L}_k\widehat{\xi}_{k/k} \tag{6.4.11c}$$

$$K_k = P_k\bar{C}_k^{\mathrm{T}}(R_k + \bar{C}_k P_k \bar{C}_k^{\mathrm{T}})^{-1} \tag{6.4.11d}$$

We now partition $\widehat{\xi}_{k/k}$ and K_k as

$$\widehat{\xi}_{k/k} = \begin{bmatrix} \widehat{x}_{k/k} \\ \widehat{x}_{k-1/k} \\ \vdots \\ \widehat{x}_{k-h/k} \end{bmatrix}, \quad K_k = \begin{bmatrix} K_{0/k} \\ K_{1/k} \\ \vdots \\ K_{h/k} \end{bmatrix} = \begin{bmatrix} P_{0,0/k}C_k^{\mathrm{T}}(R_k + C_k P_{0,0/k}C_k^{\mathrm{T}})^{-1} \\ P_{1,0/k}C_k^{\mathrm{T}}(R_k + C_k P_{0,0/k}C_k^{\mathrm{T}})^{-1} \\ \vdots \\ P_{h,0/k}C_k^{\mathrm{T}}(R_k + C_k P_{0,0/k}C_k^{\mathrm{T}})^{-1} \end{bmatrix}$$

Then, we easily obtain from (6.4.11) the $\mathbf{H}_\infty$ fixed-lag smoothing algorithm of (6.4.8). ■

For the time-invariant system, based on Theorem 6.3, we are able to obtain a solution to the infinite-horizon $\mathbf{H}_\infty$ fixed-lag smoothing problem by taking N to infinity.

Theorem 6.5: *Suppose that (C, A) is detectable and (A, B) has no uncontrollable modes on the unit circle. We assume that there exist matrices $P_{i,j}$, $\bar{P}_{i,j}$ and $\Pi_{i,j}$ $(i, j = 0, 1, \cdots, h)$ such that*

$$P_{0,0} = A\bar{P}_{0,0}A^{\mathrm{T}} + A\Pi_{0,0}A^{\mathrm{T}} + BB^{\mathrm{T}} \tag{6.4.12a}$$

$$P_{0,j} = A\bar{P}_{0,j-1} + A\Pi_{0,j-1}, \quad j \neq 0 \tag{6.4.12b}$$

$$P_{i,j} = \bar{P}_{i-1,j-1} + \Pi_{i-1,j-1}, \quad ij \neq 0 \tag{6.4.12c}$$

$$\Pi_{i,j} = \bar{P}_{i,h}L^{\mathrm{T}}(\gamma^2 I_p - L\bar{P}_{h,h}L^{\mathrm{T}})^{-1}L\bar{P}_{h,j} \tag{6.4.12d}$$

$$\bar{P}_{i,j} = P_{i,j} - P_{i,0}C^{\mathrm{T}}(R + CP_{0,0}C^{\mathrm{T}})^{-1}C^{\mathrm{T}}P_{0,j} \tag{6.4.12e}$$

$$P := \begin{bmatrix} P_{0,0} & P_{0,1} & \cdots & P_{0,h} \\ P_{1,0} & P_{1,1} & \cdots & P_{1,h} \\ \vdots & \vdots & \ddots & \vdots \\ P_{h,0} & P_{h,1} & \cdots & P_{h,h} \end{bmatrix} \geq 0 \tag{6.4.13}$$

$$\gamma^2 I_p - L\bar{P}_{h,h}L^{\mathrm{T}} > 0 \tag{6.4.14}$$

and $\bar{A}\{I_{(h+1)n} + P(\bar{C}^{\mathrm{T}}R^{-1}\bar{C} - \gamma^{-2}\bar{L}^{\mathrm{T}}\bar{L})\}^{-1}$ *has all eigenvalues inside the open unit disk.*

Then, there exists an $\mathbf{H}_\infty$ *fixed-lag smoother achieving*

$$\sup_{w,v} \frac{\|z_{k-h} - \widehat{z}_{k-h}\|_2^2}{\|w\|_2^2 + \|v\|_2^2} < \gamma^2 \tag{6.4.15}$$

In this case, one of the $\mathbf{H}_\infty$ *fixed-lag smoother is given by*

$$\widehat{x}_{k-i/k} = \widehat{x}_{k-i/k-1} + K_i(y_k - C\widehat{x}_{k/k-1}) \tag{6.4.16a}$$

$$\widehat{x}_{k+1/k} = A\widehat{x}_{k/k} \tag{6.4.16b}$$

$$\widehat{z}_{k-h} = L\widehat{x}_{k-h/k} \tag{6.4.16c}$$

$$K_i = P_{i,0}C^{\mathrm{T}}(R + CP_{0,0}C^{\mathrm{T}})^{-1} \tag{6.4.16d}$$

Remark 6.1: The necessity of the theorem is also true by Theorem 2.1.

Remark 6.2: Since the fixed-lag smoothers of (6.4.8) and (6.4.16) are derived from the central $\mathbf{H}_\infty$ filters, we call them the central $\mathbf{H}_\infty$ fixed-lag smoothers. It may be noted that the central $\mathbf{H}_\infty$ fixed-lag smoother inherits the properties of the central $\mathbf{H}_\infty$ filter in Section 6.2, because the smoother is the central $\mathbf{H}_\infty$ filter for the augmented system (6.4.2)–(6.4.4).

Remark 6.3: It is known that, for the fixed-interval smoothing problem, the $\mathbf{H}_\infty$ smoother is identical to the $\mathbf{H}_2$ optimal one (see Appendix 5.2). But the $\mathbf{H}_\infty$ fixed-lag smoother is different from the $\mathbf{H}_2$ optimal one in that $\Pi_{i,j/k}$ and $\Pi_{i,j}$ exist in (6.4.8) and (6.4.16), respectively.

5. Numerical Example

Let us consider the system given by

$$x_{k+1} = x_k + 0.1 w_k, \quad x_0 = 1$$

$$y_k = x_k + v_k, \quad z_k = x_k \; .$$

In the case where $\Pi = 1$, the optimal value of the parameter γ is $\gamma_{\rm opt} = 1$. Fig. 6.1 shows the relationship among P_k, P'_k and Q_k for $\gamma = 1.25$. We see from the figure that $P_k \geq Q_k \geq P'_k$ holds for all k.

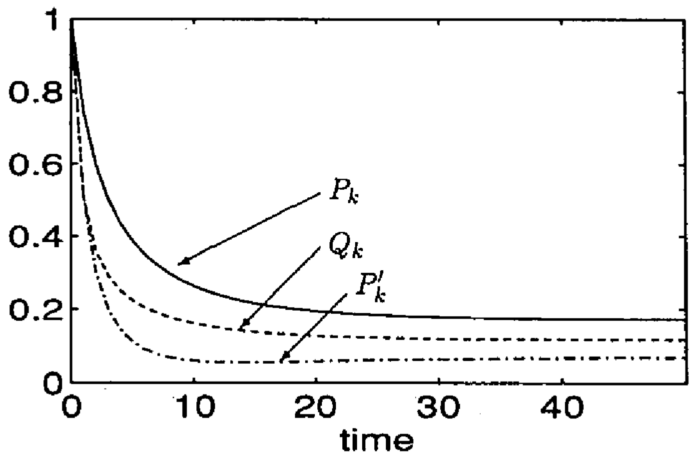
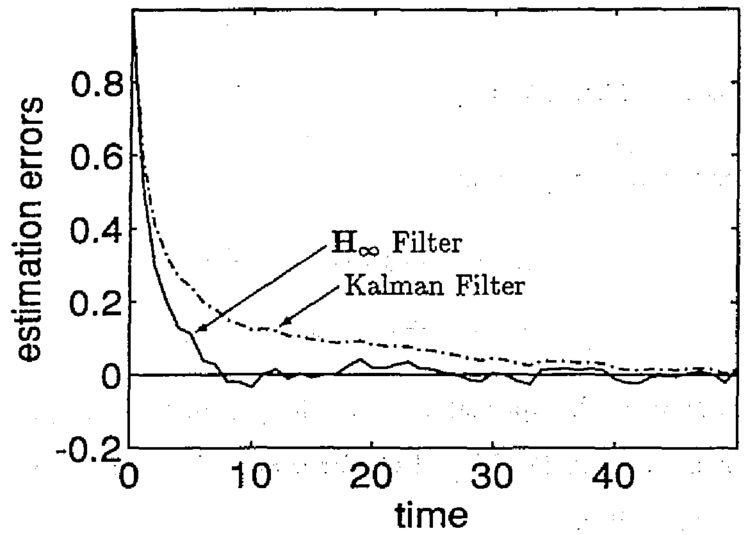


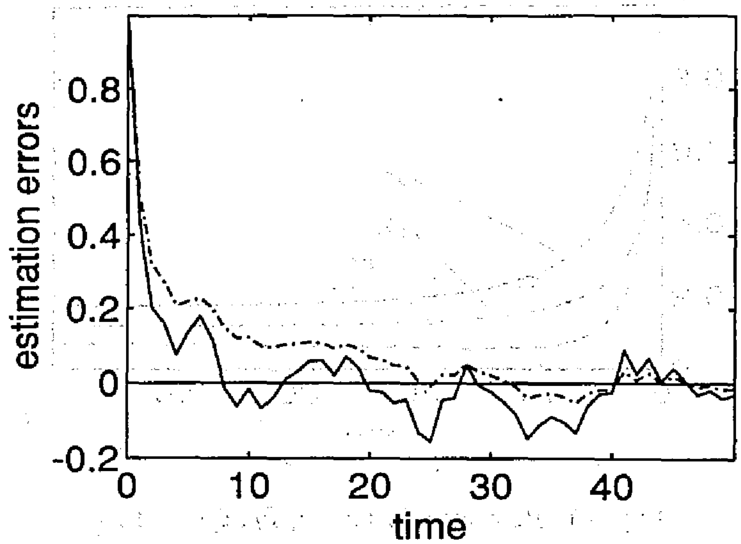
Fig. 6.1: Relationship among P_k, Q_k and P'_k

Furthermore, in order to compare the performance of the $\mathbf{H}_\infty$ filter with that of Kalman filter, we performed simulations (Fig. 6.2). In the following simulations, we set $w_k \equiv 0$ to see the sensitivity of the filters to $\nu_k = y_k - C_k \hat{x}_{k/k-1}$. The measurement noise v_k is zero mean Gaussian white noise. Figs. 6.2 (a) and (b) show the simulation results for $E\{v_k^2\} = 0.01$ and $E\{v_k^2\} = 0.09$, respectively. In the case when the measurement noise

v_k is very small, the central $\mathbf{H}_\infty$ filter presents a much better transient response than Kalman filter. On the other hand, when v_k is large, the $\mathbf{H}_\infty$ filter is more sensitive to the measurement noise than Kalman filter.



(a) $E\{v_k^2\} = 0.01$



(b) $E\{v_k^2\} = 0.09$

Fig. 6.2: Comparison of $\mathbf{H}_\infty$ filter and Kalman filter

The relationship between γ and P_k for $k = 25$ is illustrated in Fig. 6.3. We see that P_{25} is monotonically non-increasing with respect to γ. We also obtained the same results for all k.

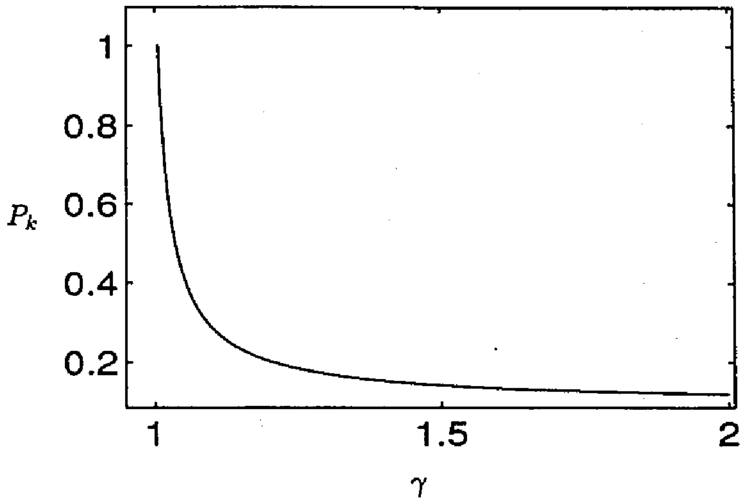


Fig. 6.3: Relationship between γ and P_k

Simulation results for various γ are shown in Fig. 6.4. The measurement noise is the zero mean white noise with $E\{v_k^2\} = 0.01$. As shown in the figure, the $\mathbf{H}_\infty$ filter becomes less sensitive to ν_k as γ becomes large.

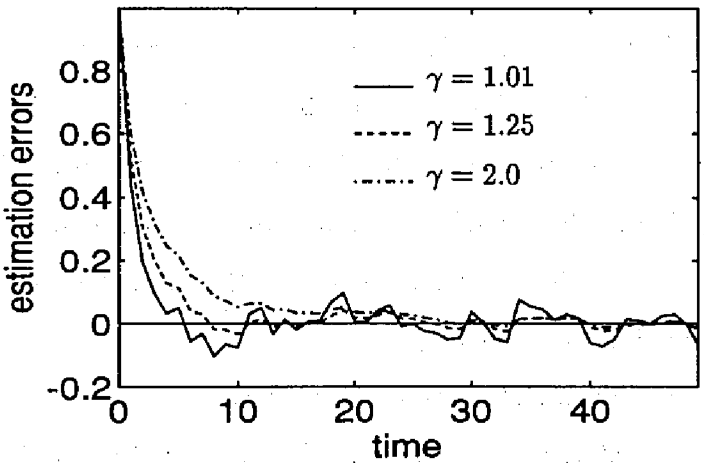


Fig. 6.4: Estimation errors for various values of γ

Next, we apply the $\mathbf{H}_\infty$ fixed-lag smoothing algorithm in Theorem 6.5 to this system. Fig. 6.5 shows the singular value plots of the transfer matrix from the disturbance $[\, w_k \; v_k \,]^{\mathrm{T}}$ to the estimation error $e_k := z_{k-h} - \widehat{z}_{k-h}$ in the case where $\gamma = 1.25$. We see from the figure that as the smoothing lag h increases, the $\mathbf{H}_\infty$ performance of the central $\mathbf{H}_\infty$ fixed-lag smoother is improved.

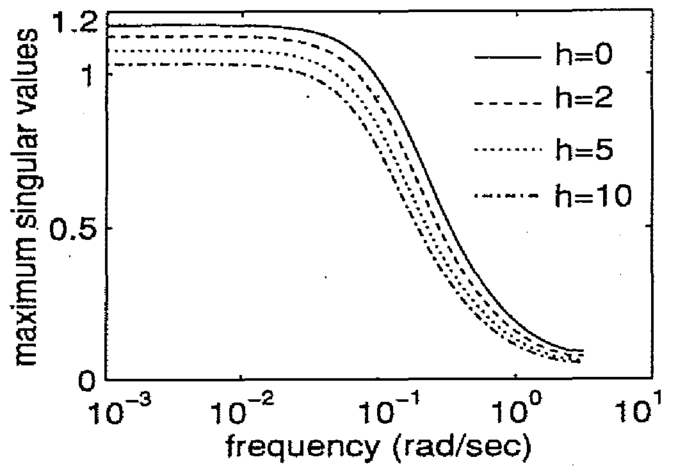


Fig. 6.5: Singular value plots of the smoothed error system

6. Concluding Remarks

In this chapter, by using Riccati difference equations, we have compared the performances of the central $\mathbf{H}_\infty$ filter and Kalman filter in the case where the disturbances are zero Gaussian white noises. The relationship between the prescribed $\mathbf{H}_\infty$ bound γ and the performance of the central $\mathbf{H}_\infty$ filter is investigated based on the monotonicity property of the $\mathbf{H}_\infty$ RDE. These results also apply to the infinite-horizon time-invariant case by replacing the RDEs with the corresponding AREs. The above results will provide a guideline for determining the value of γ.

For the time-invariant case, we have proved that the solution of the $\mathbf{H}_\infty$ RDE converges to a stabilizing solution of the corresponding ARE if the infinite-horizon $\mathbf{H}_\infty$ filtering problem is solvable.

Furthermore, we have studied the $\mathbf{H}_\infty$ fixed-lag smoothing problem by utilizing the results on the $\mathbf{H}_\infty$ filtering problem. It has been shown that the central $\mathbf{H}_\infty$ fixed-lag smoother inherits the properties of the central $\mathbf{H}_\infty$ filter, and that the $\mathbf{H}_\infty$ and $\mathbf{H}_2$ fixed-lag smoothers are different while the fixed-interval smoothers are identical.

Chapter 7

Stochastic Minimax Filtering Problem and Its Relation to $\mathbf{H}_\infty$ Filtering Problem

1. Introduction

The previous chapter has given the relationship between the $\mathbf{H}_\infty$ filtering problem and a 'deterministic' minimax filtering problem. In this chapter, we will provide an alternative game theoretic interpretation of the central $\mathbf{H}_\infty$ filter and predictor. We will consider 'stochastic' minimax state estimation problems, which are discrete-time equivalents to the problem discussed in [54]. We will derive saddle point solutions to the stochastic estimation problems based on the matrix minimum principle by Athans [1]. It is shown that the minimizer's saddle-point policies in the minimax filtering and prediction problems are identical to the central $\mathbf{H}_\infty$ filter and predictor, respectively. Therefore, the results of this chapter justify the application of the central $\mathbf{H}_\infty$ filter and predictor to the stochastic systems.

2. Stochastic Minimax Filtering Problem

2.1 Problem formulation

We first formulate the stochastic minimax filtering problem (SMFP) based on the continuous-time result of [54].

We consider a linear discrete-time system described by

$$x_{k+1} = A_k x_k + B_k w_k + s_k \tag{7.2.1}$$

$$y_k = C_k x_k + D_k v_k \tag{7.2.2}$$

where $x_k \in \mathbf{R}^n$ and $y_k \in \mathbf{R}^q$ are the state vector and the measurements, respectively. The noise disturbances $w_k \in \mathbf{R}^m$ and $v_k \in \mathbf{R}^\ell$ are mutually uncorrelated Gaussian white noises with

$$E\{w_k\} = 0, \quad E\{v_k\} = 0$$
$$E\left\{\begin{bmatrix} w_k \\ v_k \end{bmatrix}\begin{bmatrix} w_\tau \\ v_\tau \end{bmatrix}^{\mathrm{T}}\right\} = \begin{bmatrix} I_m & 0 \\ 0 & I_\ell \end{bmatrix}\delta_{k\tau}$$

where δ_{ij} denotes the Kronekar's delta. The initial state x_0 is generated by the Gaussian distribution $\mathcal{N}(\bar{x}_0, \Pi)$. We also assume that $D_k \in \mathbf{R}^{q\times\ell}$ is right invertible and we define $R_k := D_k D_k^{\mathrm{T}}$. The disturbance $s_k \in \mathbf{R}^n$ is to be defined below.

Let $\widehat{x}_{k/t}$ be an estimate of x_k based on the measurement set $\{y_0, \cdots, y_t\}$. To guarantee the unbiasedness of the estimate $\widehat{x}_{k/k}$, we assume that $\widehat{x}_{k/k}$ is generated by the following filtering algorithm:

$$\widehat{x}_{k/k} = \widehat{x}_{k/k-1} + K_k(y_k - C_k\widehat{x}_{k/k-1}) \tag{7.2.3}$$

$$\widehat{x}_{k+1/k} = A_k\widehat{x}_{k/k}, \quad \widehat{x}_{0/-1} = \bar{x}_0 \tag{7.2.4}$$

where K_k is the filter gain to be determined.

In this minimax problem, we introduce the fictitious output $z_k \in \mathbf{R}^p$ defined by

$$z_k = L_k x_k \tag{7.2.5}$$

The estimation at time k is disturbed by the disturbance s_k defined by

$$s_k = M_k\left(\frac{e_k}{\gamma} + n_k\right) \tag{7.2.6}$$

where $e_k = z_k - L_k\hat{x}_{k/k}$ and n_k is the Gaussian white noise with $E\{n_k\} = 0$, $E\{n_k n_\tau^{\mathrm{T}}\} = I_n\delta_{k\tau}$. We assume that w_k, v_k and n_k are mutually uncorrelated. The block diagram of the SMFP is illustrated in Fig. 7.1.

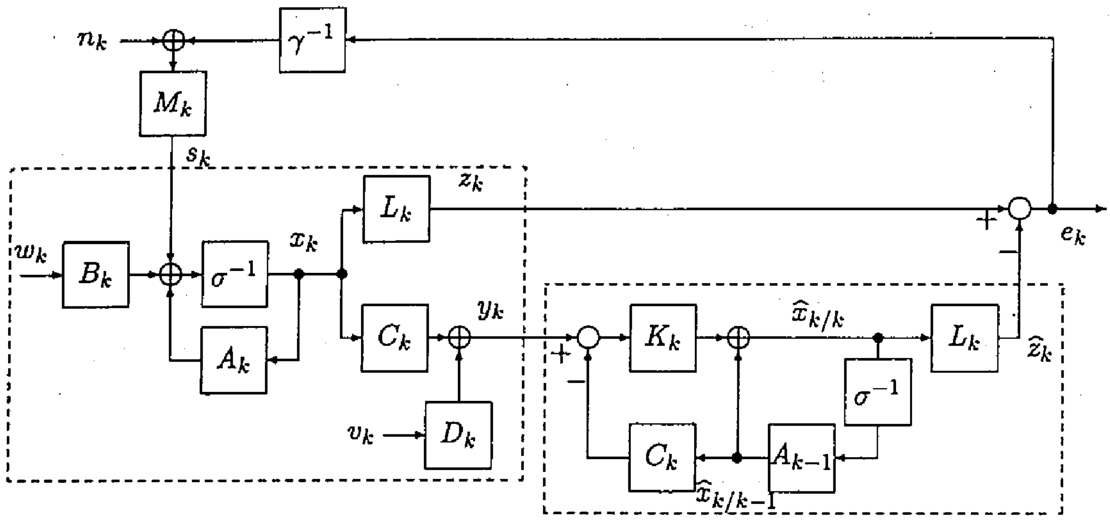


Fig. 7.1: Diagram of the stochastic minimax filtering problem

We first introduce a cost function as follows. Define the state estimation error $\tilde{x}_k := x_k - \hat{x}_{k/k}$. Then from (7.2.1)–(7.2.6), we obtain

$$\tilde{x}_{k+1} = (I_n - K_{k+1}C_{k+1})\left\{\left(A_k + \frac{M_kL_k}{\gamma}\right)\tilde{x}_k + B_kw_k + M_kn_k\right\} - K_{k+1}D_{k+1}v_{k+1}$$

$$\tilde{x}_0 = x_0 - \hat{x}_{0/0} = (I_n - K_0C_0)(x_0 - \bar{x}_0) + K_0D_0v_0$$

Then $\tilde{x}_k$ is decomposed as

$$\tilde{x}_k = \tilde{x}_{1,k} + \tilde{x}_{2,k}, \quad \tilde{x}_{1,0} = x_0 - \hat{x}_{0/0}, \ \tilde{x}_{2,0} = 0 \tag{7.2.7}$$

$$\tilde{x}_{1,k+1} = (I_n - K_{k+1}C_{k+1})\left\{\left(A_k + \frac{M_kL_k}{\gamma}\right)\tilde{x}_{1,k} + B_kw_k\right\} - K_{k+1}D_{k+1}v_{k+1} \tag{7.2.8}$$

$$\tilde{x}_{2,k+1} = (I_n - K_{k+1}C_{k+1})\left\{\left(A_k + \frac{M_kL_k}{\gamma}\right)\tilde{x}_{2,k} + M_kn_k\right\} \tag{7.2.9}$$

where $\tilde{x}_{1,k}$ is the state estimation error due to w_k and v_k, and $\tilde{x}_{2,k}$ is the error due to n_k. Arbitrarily large norm values of M_k increases $\|\tilde{x}_{2,k}\|$ and $\|e_k\|$ arbitrarily, and thus a cost function must contain a penalty imposed on M_k in order to limit the range that M_k may attain. We therefore define the following cost function as

$$J_N(K,M) = \sum_{k=0}^{N} E\{\tilde{x}_{1,k}^{\mathrm{T}} W_k \tilde{x}_{1,k} - \tilde{x}_{2,k}^{\mathrm{T}} W_k \tilde{x}_{2,k}\} \tag{7.2.10}$$

where W_k is a positive definite symmetric bounded matrix, and $K = \{K_0, \cdots, K_N\}$ and $M = \{M_0, \cdots, M_N\}$. This minimax filtering problem can be interpreted as the dynamic minimax game with a quadratic cost. The minimizer adjusts K_k so that $J_N(K,M)$ is minimized, while the maximizer adjusts M_k so that $J_N(K,M)$ is maximized.

Lemma 7.1: *The cost function of (7.2.10) is expressed as*

$$J_N = \mathrm{Tr}\{W_N \bar{P}_N\} + \sum_{k=0}^{N-1} \mathrm{Tr}\{W_k \bar{P}_k\} \tag{7.2.11}$$

where

$$\bar{P}_k = (I_n - K_k C_k) P_k (I_n - K_k C_k)^{\mathrm{T}} + K_k R_k K_k^{\mathrm{T}} \tag{7.2.12}$$

$$P_{k+1} = \left(A_k + \frac{M_k L_k}{\gamma}\right) \bar{P}_k \left(A_k + \frac{M_k L_k}{\gamma}\right)^{\mathrm{T}} + B_k B_k^{\mathrm{T}} - M_k M_k^{\mathrm{T}}, \quad P_0 = \Pi \tag{7.2.13}$$

Proof: We define $P_{1,k} = E\{\tilde{x}_{1,k}\tilde{x}_{1,k}^{\mathrm{T}}\}$ and $P_{2,k} = E\{\tilde{x}_{2,k}\tilde{x}_{2,k}^{\mathrm{T}}\}$. Since w_k, v_k and n_k are the uncorrelated white noise, we get

$$\begin{aligned} P_{1,k+1} &= \left\{(I_n - K_{k+1}C_{k+1})\left(A_k + \frac{M_k L_k}{\gamma}\right)\right\} P_{1,k} \left\{(I_n - K_{k+1}C_{k+1})\left(A_k + \frac{M_k L_k}{\gamma}\right)\right\}^{\mathrm{T}} \\ &\quad + (I_n - K_{k+1}C_{k+1}) B_k B_k^{\mathrm{T}} (I_n - K_{k+1}C_{k+1})^{\mathrm{T}} + K_{k+1} R_{k+1} K_{k+1}^{\mathrm{T}} \end{aligned} \tag{7.2.14}$$

$$\begin{aligned} P_{2,k+1} &= \left\{(I_n - K_{k+1}C_{k+1})\left(A_k + \frac{M_k L_k}{\gamma}\right)\right\} P_{2,k} \left\{(I_n - K_{k+1}C_{k+1})\left(A_k + \frac{M_k L_k}{\gamma}\right)\right\}^{\mathrm{T}} \\ &\quad + (I_n - K_{k+1}C_{k+1}) M_k M_k^{\mathrm{T}} (I_n - K_{k+1}C_{k+1})^{\mathrm{T}} \end{aligned} \tag{7.2.15}$$

$$P_{1,0} = (I_n - K_0 C_0)\Pi(I_n - K_0 C_0)^{\mathrm{T}} + K_0 R_0 K_0^{\mathrm{T}}, \quad P_{2,0} = 0 \tag{7.2.16}$$

We define $\bar{P}_k = P_{1,k} - P_{2,k}$ and

$$P_{k+1} = \left(A_k + \frac{M_k L_k}{\gamma}\right) \bar{P}_k \left(A_k + \frac{M_k L_k}{\gamma}\right)^{\mathrm{T}} + B_k B_k^{\mathrm{T}} - M_k M_k^{\mathrm{T}}, \quad P_0 = \Pi$$

Then, from the definition of J_N, we get

$$J_N = \sum_{k=0}^{N} \mathrm{Tr}\{W_k \bar{P}_k\} = \mathrm{Tr}\{W_N \bar{P}_N\} + \sum_{k=0}^{N-1} \mathrm{Tr}\{W_k \bar{P}_k\}$$

Furthermore, (7.2.12) directly follows from (7.2.14) and (7.2.15). ■

2.2 Saddle point solution

In this chapter, we wish to find a saddle point solution to the SMFP and the corresponding saddle point policies $K^* = \{K_0^*, \cdots, K_N^*\}$ and $M^* = \{M_0^*, \cdots, K_N^*\}$. The saddle point solution is defined by

$$J_N(K^*, M) \le J_N(K^*, M^*) \le J_N(K, M^*) \tag{7.2.17}$$

We derive a saddle point solution of this minimax problem by adopting the matrix minimum principle [1]. To perform the optimization with respect to K and M, we form the Hamiltonian

$$H_k = \mathrm{Tr}\{W_k P_k\} + \mathrm{Tr}\{(P_{k+1} - P_k)\Lambda_{k+1}^{\mathrm{T}}\} \tag{7.2.18}$$

where Λ_t is the costate matrix. Let P_k^*, $\bar{P}_k^*$ and Λ_k^* be the values of P_k, $\bar{P}_k$ and Λ_k corresponding to the saddle point policy (K^*, M^*), respectively. Then, we can obtain the saddle point policies by solving the following necessary conditions:

$$\Lambda_{k+1}^* - \Lambda_k^* = -\left.\frac{\partial H_k}{\partial P_k}\right|_{(K,M)=(K^*,M^*)}, \quad \Lambda_N^* = W_N \tag{7.2.19}$$

$$H_k(K_{k+1}^*, M_k) \le H_k(K_{k+1}, M_k) \quad \forall K_{k+1} \in \mathbf{R}^{n\times q} \tag{7.2.20}$$

$$H_k(K_{k+1}, M_k) \le H_k(K_{k+1}, M_k^*) \quad \forall M_k \in \mathbf{R}^{n\times p} \tag{7.2.21}$$

It may be noted that the derivative formulae for trace of matrices in [1] are very useful for solving the condition of (7.2.19). We see from (7.2.19) that

$$\Lambda_k^* = \left\{(I_n - K_{k+1}^* C_{k+1})(A_k + \frac{M_k^* L_k}{\gamma})\right\} \Lambda_{k+1}^* \left\{(I_n - K_{k+1}^* C_{k+1})(A_k + \frac{M_k^* L_k}{\gamma})\right\}^{\mathrm{T}} + W_k \tag{7.2.22}$$

Since W_k is positive definite and symmetric, so is Λ_k^*. Note that P_k and $\bar{P}_k$ can be expressed as

$$\bar{P}_k = K_k(R_k + C_k P_k C_k^{\mathrm{T}})K_k^{\mathrm{T}} - K_k C_k P_k - P_k C_k^{\mathrm{T}} K_k^{\mathrm{T}} + P_k \tag{7.2.23}$$

$$\begin{aligned} P_{k+1} = &-\gamma^{-2} M_k(\gamma^2 I_p - L_k \bar{P}_k L_k^{\mathrm{T}})M_k^{\mathrm{T}} + A_k \bar{P}_k A_k^{\mathrm{T}} + B_k B_k^{\mathrm{T}} \\ &+\gamma^{-1} M_k L_k \bar{P}_k A_k^{\mathrm{T}} + \gamma^{-1} A_k \bar{P}_k L_k^{\mathrm{T}} M_k^{\mathrm{T}} \end{aligned} \tag{7.2.24}$$

Since Λ_k^* is positive definite and $\bar{P}_k$ is independent of K_{k+1}, we see from (7.2.23) that there exists a unique K_{k+1}^* satisfying (7.2.20) if and only if

$$V_{k+1} := R_{k+1} + C_{k+1} P_{k+1}^* C_{k+1}^{\mathrm{T}} > 0 \tag{7.2.25}$$

We now assume $P_{k+1} = P_{k+1}^*$. Then, the square completion of $\bar{P}_{k+1}$ with respect to K_{k+1} yields

$$\begin{aligned}\bar{P}_{k+1} = (K_{k+1} - P_{k+1}^* C_{k+1}^{\mathrm{T}} V_{k+1}^{-1}) V_{k+1} (K_{k+1} - P_{k+1}^* C_{k+1}^{\mathrm{T}} V_{k+1}^{-1})^{\mathrm{T}} \\ + P_{k+1}^* - P_{k+1}^* C_{k+1}^{\mathrm{T}} V_{k+1}^{-1} C_{k+1} P_{k+1}^*\end{aligned}$$

Thus, (7.2.20) is satisfied by taking $K_{k+1}^* = P_{k+1}^* C_{k+1}^{\mathrm{T}} V_{k+1}^{-1}$. Similarly, there exists a unique matrix M_k^* satisfying (7.2.21) if and only if

$$\Psi_k := \gamma^2 I_p - L_k \bar{P}_k^* L_k^{\mathrm{T}} > 0 \tag{7.2.26}$$

In this case, M_k^* satisfying (7.2.21) is given by $M_k^* = \gamma A_k \bar{P}_k^* L_k^{\mathrm{T}} \Psi_k^{-1}$.

Substituting $P_k = P_k^*$, $\bar{P}_k = \bar{P}_k^*$, $K_k = K_k^*$ and $M_k = M_k^*$ into (7.2.12) and (7.2.13) yields

$$P_{k+1}^* = A_k P_k^* \Sigma_k^{-1} A_k^{\mathrm{T}} + B_k B_k^{\mathrm{T}}, \qquad P_0^* = \Pi \tag{7.2.27a}$$

$$\Sigma_k = I_n + (C_k^{\mathrm{T}} R_k^{-1} C_k - \gamma^{-2} L_k^{\mathrm{T}} L_k) P_k^* \tag{7.2.27b}$$

and

$$\bar{P}_k^* = P_k^* - P_k^* C_k^{\mathrm{T}} V_k^{-1} C_k P_k^* = P_k^* (I_n + C_k^{\mathrm{T}} R_k^{-1} C_k P_k^*)^{-1} \tag{7.2.28}$$

It is easily seen from Lemma 5.1 that P_k^* is positive semi-definite if $\Psi_k > 0$ for all k.

The equation (7.2.27) is the well-known $\mathbf{H}_\infty$ RDE. Hence, the minimizer's optimal policy K^* provides the $\mathbf{H}_\infty$ filtering algorithm.

The following theorem summarizes the above discussions.

Theorem 7.1: *Suppose that there exists a unique saddle point solution to the SMFP. Then, there exists a positive semi-definite solution to the RDE (7.2.27) such that $\Psi_k > 0$ holds over the interval $[0, N]$.*

We next show that the existence of such a solution to the RDE (7.2.27) is also sufficient for the existence of a saddle point solution.

Theorem 7.2: *Suppose that there exists a positive semi-definite solution to the RDE (7.2.27) satisfying $\Psi_k > 0$. Then, there exists a saddle point solution, and the corresponding saddle point policies are given by*

$$K_k^* = P_k^* C_k^{\mathrm{T}} (R_k + C_k P_k^* C_k^{\mathrm{T}})^{-1} \tag{7.2.29}$$

$$M_k^* = \gamma A_k \bar{P}_k^* L_k^{\mathrm{T}} (\gamma^2 I_p - L_k \bar{P}_k^* L_k^{\mathrm{T}})^{-1} \tag{7.2.30}$$

Proof: Define K_k^* and M_k^* by (7.2.29) and (7.2.30). Then, we easily see that

$$\bar{P}_k^* = (I_n - K_k C_k) P_k^* (I_n - K_k C_k)^{\mathrm{T}} + K_k R_k K_k^{\mathrm{T}} - \widetilde{K}_k V_k^{-1} \widetilde{K}_k^{\mathrm{T}} \tag{7.2.31}$$

$$\begin{aligned} P_{k+1}^* = {} & \left(A_k + \frac{M_k L_k}{\gamma}\right) \bar{P}_k^* \left(A_k + \frac{M_k L_k}{\gamma}\right)^{\mathrm{T}} + B_k B_k^{\mathrm{T}} - M_k M_k^{\mathrm{T}} \\ & + \gamma^{-2} \widetilde{M}_k \Psi_k^{-1} \widetilde{M}_k^{\mathrm{T}} \end{aligned} \tag{7.2.32}$$

where $\widetilde{K}_k = K_k - K_k^*$ and $\widetilde{M}_k = M_k - M_k^*$. We also define $\Delta P_k = P_k^* - P_k$ and $\Delta \bar{P}_k = \bar{P}_k^* - \bar{P}_k$. Subtracting (7.2.12) from (7.2.31) yields

$$\Delta \bar{P}_k = (I_n - K_k C_k)(\Delta P_k)(I_n - K_k C_k)^{\mathrm{T}} - \widetilde{K}_k V_k^{-1} \widetilde{K}_k^{\mathrm{T}} \tag{7.2.33}$$

Similarly, it follows from (7.2.13) and (7.2.32) that

$$\Delta P_{k+1} = \left(A_k + \frac{M_k L_k}{\gamma}\right) (\Delta \bar{P}_k) \left(A_k + \frac{M_k L_k}{\gamma}\right)^{\mathrm{T}} + \gamma^{-2} \widetilde{M}_k \Psi_k^{-1} \widetilde{M}_k^{\mathrm{T}} \tag{7.2.34}$$

First, we assume $K_k = K_k^*$. Then, (7.2.33) reduces to

$$\Delta \bar{P}_k = (I_n - K_k^* C_k)(\Delta P_k)(I_n - K_k^* C_k)^{\mathrm{T}} \tag{7.2.35}$$

Since $P_0 = P_0^* = \Pi$, we get $\Delta P_0 = 0$ and $\Delta \bar{P}_0 \geq 0$. We assume that $\Delta \bar{P}_k \geq 0$ holds for $k = 0, \cdots, t$. It then follows from (7.2.34),(7.2.35) that $\Delta P_{t+1} \geq 0$ and $\Delta \bar{P}_{t+1} \geq 0$ since $\Psi_t > 0$. Hence, $\Delta \bar{P}_k \geq 0$ holds for any $k \geq 0$ by induction. Thus, we obtain $J_N(K^*, M) \leq J_N(K^*, M^*)$.

Next, we assume $M_k = M_k^*$ for all k. It follows from (7.2.32) that

$$\Delta P_{k+1} = \left(A_k + \frac{M_k^* L_k}{\gamma}\right) (\Delta \bar{P}_k) \left(A_k + \frac{M_k^* L_k}{\gamma}\right)^{\mathrm{T}} \tag{7.2.36}$$

Thus, similarly to the case where $K_k = K_k^*$, we can prove $\Delta P_k \le 0$ for all k. This implies $J_N(K^*, M^*) \le J_N(K, M^*)$.

Consequently, we have proved that the policies of (7.2.29) and (7.2.30) are the saddle point policies satisfying (7.2.17) for the SMFP. ■

3. Stochastic Minimax Prediction Problem

We next consider the stochastic minimax prediction problem (SMPP) where $\{y_0, \cdots, y_{k-1}\}$ rather than $\{y_0, \cdots, y_{k-1}, y_k\}$ is available for the estimation at time k. The block diagram of the SMPP is illustrated in Fig. 7.2. The argument in the filtering case is still valid with slight modifications. In the prediction case, the state estimate of x_k is generated by the following one-step prediction algorithm:

$$\widehat{x}_{k+1/k} = A_k\widehat{x}_{k/k-1} + K_k(y_k - C_k\widehat{x}_{k/k-1}), \quad \widehat{x}_{0/-1} = \bar{x}_0 \tag{7.3.1}$$

Since y_k is not available at time k, the disturbance s_k is also defined using the one-step prediction error:

$$s_k = M_k\left(\frac{z_k - L_k\widehat{x}_{k/k-1}}{\gamma} + n_k\right) \tag{7.3.2}$$

where n_k is the Gaussian white noise with $E\{n_k\} = 0$ and $E\{n_k n_\tau^{\mathrm{T}}\} = I_n\delta_{k\tau}$.

We here define the state estimation error by $\widetilde{x}_k := x_k - \widehat{x}_{k/k-1}$. Then, $\widetilde{x}_k$ is decomposed as

$$\widetilde{x}_k = \widetilde{x}_{1,k} + \widetilde{x}_{2,k}, \qquad \widetilde{x}_{1,0} = x_0 - \bar{x}_0, \ \widetilde{x}_{2,0} = 0 \tag{7.3.3}$$

$$\widetilde{x}_{1,k+1} = \left(A_k + \frac{M_kL_k}{\gamma} - K_kC_k\right)\widetilde{x}_{1,k} + B_kw_k - K_kD_kv_k \tag{7.3.4}$$

$$\widetilde{x}_{2,k+1} = \left(A_k + \frac{M_kL_k}{\gamma} - K_kC_k\right)\widetilde{x}_{2,k} + M_kn_k \tag{7.3.5}$$

We define the cost function by

$$\begin{aligned} J_N(K, L) &= \sum_{k=0}^{N} E\{\widetilde{x}_{1,k}^{\mathrm{T}} W_k \widetilde{x}_{1,k} - \widetilde{x}_{2,k}^{\mathrm{T}} W_k \widetilde{x}_{2,k}\} \\ &= \mathrm{Tr}\{W_N P_N\} + \sum_{k=0}^{N-1} \mathrm{Tr}\{W_k P_k\} \end{aligned}$$

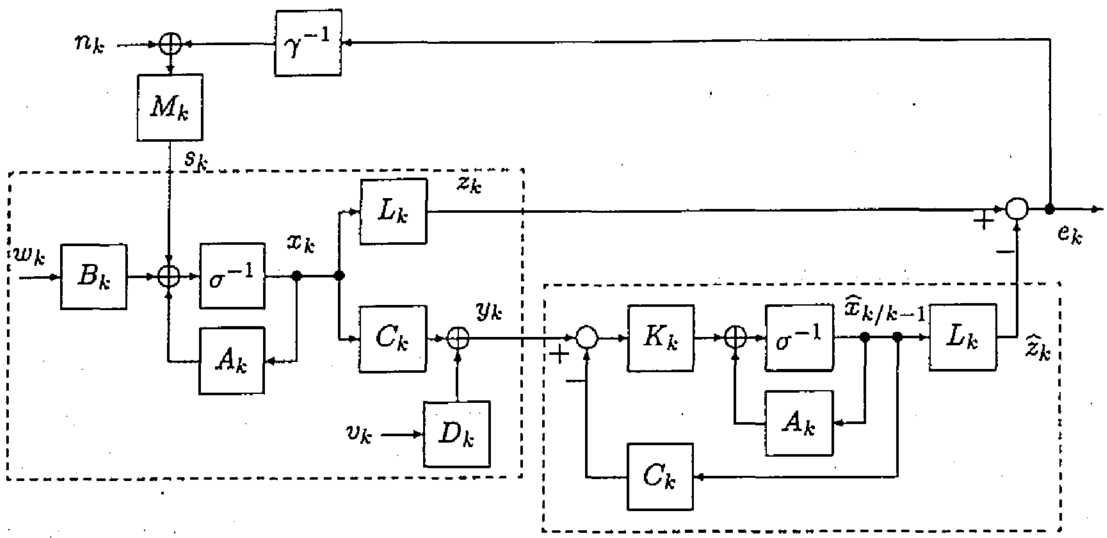


Fig. 7.2: Diagram of the stochastic minimax prediction problem

where $P_k = E\{\widetilde{x}_{1,k}\widetilde{x}_{1,k}^{\mathrm{T}}\} - E\{\widetilde{x}_{2,k}\widetilde{x}_{2,k}^{\mathrm{T}}\}$. Since w_k, v_k and n_k are mutually uncorrelated, it is easy to verify that

$$P_{k+1} = \left(A_k + \frac{M_k L_k}{\gamma} - K_k C_k\right) P_k \left(A_k + \frac{M_k L_k}{\gamma} - K_k C_k\right)^{\mathrm{T}}$$
$$+B_k B_k^{\mathrm{T}} + K_k R_k K_k^{\mathrm{T}} - M_k M_k^{\mathrm{T}}$$

Similarly to the filtering case, we obtain a saddle point solution to the SMPP by applying the matrix minimum principle.

Theorem 7.3: *There exists a unique saddle-point solution of the SMPP defined above if and only if there exists a positive semi-definite solution to the RDE (7.2.27) such that* $\gamma^2 I_p - L_k P_k^* L_k^{\mathrm{T}} > 0$ *holds. In this case, the saddle point policies are given by*

$$K_k^* = A_k \widehat{P}_k^* C_k^{\mathrm{T}} (R_k + C_k \widehat{P}_k^* C_k^{\mathrm{T}})^{-1} = A_k P_k^* \Sigma_k^{-1} C_k^{\mathrm{T}} R_k^{-1} \tag{7.3.6}$$

$$M_k^* = \gamma A_k P_k^* \Sigma_k^{-1} L_k^{\mathrm{T}} \tag{7.3.7}$$

where $\widehat{P}_k^* = P_k^*(I_n - \gamma^{-2} L_k^{\mathrm{T}} L_k P_k^*)^{-1}$.

It may be noted that the minimizer's optimal policy (7.3.6) provides the $\mathbf{H}_\infty$ prediction algorithm derived in Chapter 5.

4. Concluding Remarks

In this chapter, we discussed the existence of a saddle point solution to the stochastic minimax filtering and prediction problems. It has turned out that the estimators' saddle point policies provide the central $\mathbf{H}_\infty$ filter and predictor. These results obtained in this chapter provide alternative interpretations of these $\mathbf{H}_\infty$ state estimators, and hence justify the application of them to stochastic systems.

Chapter 8

Conclusion

This thesis has considered the solutions to the $\mathbf{H}_\infty$ filtering problems and their analyses. The main contributions of the thesis are summarized as follows.

In **Chapter 2**, we have given a solvability condition of the $\mathbf{H}_\infty$ filtering problem based on the model matching approach using (J, J')-spectral factorization for the infinite-horizon time-invariant case. We have also derived a complete parametrization of all solutions to the $\mathbf{H}_\infty$ filtering problem. The pole-zero cancellation structure of the $\mathbf{H}_\infty$ filtering problem has shown by using the chain scattering representation. Furthermore, we have given a solution to the $\mathbf{H}_\infty$ prediction problem as a special case of the $\mathbf{H}_\infty$ filtering problem. The present approach provides a unified solution to the $\mathbf{H}_\infty$ filtering and prediction problem.

Similarly to the $\mathbf{H}_\infty$ control case, the free parameter contained in the parametrization of $\mathbf{H}_\infty$ filters can be used for achieving an additional design specification, e.g. $\mathbf{H}_2$ performance, as well as the $\mathbf{H}_\infty$ error bound. In **Chapter 3**, we have proposed a method for adjusting the free parameter so that the constraints on the unit circle in the complex plane are satisfied. By the proposed method, we can reject the undesirable effects due to the step or periodic disturbances. We have also shown the relationship between the state-space model of the disturbance and the structure of the proposed $\mathbf{H}_\infty$ filter when the disturbance is a step function.

In **Chapter 4**, we have examined the behavior of the stabilizing solution of the $\mathbf{H}_\infty$ ARE with respect to the variation of the prescribed $\mathbf{H}_\infty$ norm bound γ. The infimum of the parameter γ, for which a stabilizing solution to the $\mathbf{H}_\infty$ ARE exists, is characterized in terms of the $\mathbf{L}_\infty$ norm of a certain transfer matrix. The stabilizing solution is a monotonically non-increasing convex function of γ. A new parametrization of all $\mathbf{H}_\infty$ filter has been also derived. Based on the above results, we have shown that the size of the set of all $\mathbf{H}_\infty$ filters is monotonically increasing with respect to γ, and that there are possibilities that the degree of freedom of the $\mathbf{H}_\infty$ filter decreases at the optimum. We have also developed an $\mathbf{H}_2/\mathbf{H}_\infty$ filtering algorithm which makes use of the free parameter contained in the new parametrization of $\mathbf{H}_\infty$ filters. The present results provide a guideline for selecting the values of the parameters γ and the free parameter. It may be also noted that the analyses in this paper can be applied to those of the $\mathbf{H}_\infty$ controllers for 2-block problems.

Chapter 5 has shown that the solutions to the minimax filtering and predictions problems are given by the central $\mathbf{H}_\infty$ filter and one-step predictor, respectively. Furthermore, in deducing the minimax solutions, we have derived the worst-case disturbances in the sense that they maximize the cost function; in other words, they maximize the energy gain between the estimation error and the disturbances. For the infinite-horizon time-varying case, in addition to the existence of a solution to the $\mathbf{H}_\infty$ RDE, it is required that the Riccati solution is a stabilizing solution.

In **Chapter 6**, by using RDEs, we have compared the performances of the $\mathbf{H}_\infty$ filter and Kalman filter for the finite-horizon filtering problem where the disturbances are Gaussian white noises. The relation between the prescribed $\mathbf{H}_\infty$ bound γ and the performance of the central $\mathbf{H}_\infty$ filter is examined based on the monotonicity of the $\mathbf{H}_\infty$ RDE. These results also apply to the infinite-horizon time-invariant case by replacing the RDEs with the corresponding AREs. In the time-invariant case, we have also given a sufficient condition for the convergence of the solution of $\mathbf{H}_\infty$ RDE. The above results in this chapter provide a guideline for determining the value of γ. Furthermore, we have derived a solution to the $\mathbf{H}_\infty$ fixed-lag smoothing problem by reducing the fixed-lag smoothing problem to a filtering problem.

In **Chapter 7**, we have discussed the existence of a saddle point solution to the stochastic minimax filtering and prediction problems. It has turned out that the minimizer's saddle point policies generate the $\mathbf{H}_\infty$ filter and predictor. These results provide alternative interpretations of these $\mathbf{H}_\infty$ state estimators, and provide a justification of the application of the $\mathbf{H}_\infty$ state estimators to the stochastic system with unknown disturbance statistics.

In the following of this chapter, we discuss the direction of the future research. Throughout this thesis, we have considered the state estimation of the system with unknown disturbance statistics while its dynamics is exactly known. Recently, the robust filtering for the system with structured uncertain dynamics has been solved via the quadratic stabilization technique, where the $\mathbf{H}_\infty$ control problem of an uncertain system is reduced to a certain $\mathbf{H}_\infty$ control problem of a system without uncertainty [50]. This approach however does not make clear the relation between the resulting robust $\mathbf{H}_\infty$ filter and the $\mathbf{H}_\infty$ filter for the nominal system. The results in Chapters 2–4 can be extended to clarify this relationship, and provide an insight into the robustness analyses of the nominal $\mathbf{H}_\infty$ filter.

For time-varying systems, the connection between LMS (Least Mean Square) and $\mathbf{H}_\infty$ filtering algorithms has been studied by Hassibi *et al.* [21]. They have justified the application of the LMS adaptive filtering algorithm theoretically by showing its $\mathbf{H}_\infty$ optimality. Their results indicate that the LMS algorithm inherits the properties of the central $\mathbf{H}_\infty$ filter given in Chapter 6, and that the results in Chapter 6 can be extended to the analyses of the LMS adaptive filter.

Very few application examples of the $\mathbf{H}_\infty$ filtering algorithm have been reported although the solution to the $\mathbf{H}_\infty$ filtering problem has been almost established. One of the most important direction of the future research is the applications of the $\mathbf{H}_\infty$ filtering algorithm to practical problems. The author hopes that the results in this thesis will be a help for the practical consideration of the $\mathbf{H}_\infty$ filter.

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Publications

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Chapter 2

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Chapter 3

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